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On Subsonic, Supersonic and Hypersonic Inflectional-Wave/Vortex Interaction

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Abstract. An unstable inflection point developing in an oncoming two-dimensional boundary layer can give rise to nonlinear three-dimensional inflectional-wave/vortex interaction as described in recent papers by Hall and Smith [1], Brown *et al.* [2], and Smith *et al.* [3]. In the current study on the compressible range the flow is examined theoretically just downstream of the linear neutral position, in order to understand how the interaction may be initiated. The research addresses both moderately and strongly compressible regimes. In the latter regime the vorticity mode, the most dangerous one, is taken as the wave part, causing the hypersonic interaction to become concentrated in a thin temperature-adjustment layer lying at the outer edge of the boundary layer, just below the free stream. In both regimes, the result is a nonlinear integro-differential equation for the wave-pressure which implies four different types of downstream behaviour for the interaction – a far-downstream saturation, a finite-distance singularity, exponentially decaying waves (leaving pure vortex motion) or periodicity. In a principal finding of the study, the coefficients of the equation are worked out explicitly for hypersonic flow, and in particular for the case of unit Prandtl number and a Chapman fluid, where it is shown that for sufficiently high wall temperatures the wave angle of propagation must lie between 45° and 90° relative to the free-stream direction and also no periodic solutions may occur then. The theory applies also to wake flows and others. Connections with experimental findings are noted.

1. Introduction

In practical configurations, including boundary layers, free shear layers and wakes, transition of compressible-fluid flows is a very significant phenomenon indeed with regard to the performance of various transonic, supersonic and hypersonic flight vehicles and turbine blades as well as to designing tunnel tests. See for example the review on hypersonic flow by Townend [4]. Experiments in the hypersonic range in particular point to transition originating at the outer edge of the boundary layer, a finding which is of much interest. On the theoretical side, however, comparatively little is known on compressible transition, especially for the hypersonic range, beyond linear stability theory and corresponding $\exp(N)$ methods. Our concern here is to attempt first to advance theoretical understanding of the necessarily nonlinear processes involved in transition, and specifically to focus attention on what appears to be one of the most readily encountered transition paths, occurring at remarkably low amplitudes and associated with vortex/wave interaction. Second, and in consequence, it is found that explicit results are obtainable in the hypersonic range, in contrast with lower Mach numbers. Third, a possible theoretical explanation seems to emerge for the experimental finding above in the hypersonic range.

Vortex/wave interaction theory is one of the most recent nonlinear theories aimed at increasing insight and possible modelling of a flow's transition from a laminar to a turbulent state at large Reynolds numbers. It follows the theories of pressure-displacement interaction (e.g. [15–18]) and nonlinear Euler interaction (e.g. [19,20]) and as with these theories it is of a rational nature. Vortex/wave interactions arise in instability and transition when a small-amplitude three-dimensional wave disturbance self interacts, under the influence of inertia, to induce a mean vortical component and this vortical component is sufficiently large to alter nonlinearly the erstwhile linear evolution of the wave. Typically the wave part has relatively short time and length scales whereas the vortex part, composed of longitudinal streamwise vortices, has longer length scales in general. Vortex/wave interaction is termed weak or strong according as to whether the vortical mean-flow correction is small or of order unity, respectively, relative to the original basic flow.

Vortex/wave interaction theory has been developed for two main types of wave, namely viscous Tollmien–Schlichting and inviscid Rayleigh-type inflectional waves. More attention has been directed towards the former, see e.g. refs. [21–28]. However, vortex/inflectional-wave interactions are seemingly more significant in the sense that they can be triggered by considerably smaller disturbances than in the counterpart Tollmien–Schlichting case. In view of this significant factor, a rational explanation of vortex/inflectional-wave interaction has been sought, starting with Hall and Smith's [1] findings for compressible boundary layers (based on ideas in the above papers). They demonstrated that much of the interaction is essentially concentrated in the relatively thin critical layer, i.e. the region where the near-neutral wave's behaviour is adjusted by viscous forces from one side of the generalised inflection point to the other. This is due in part to the singular nature of the wave-amplitude at the approach to the critical layer, leading to an amplified wave-forcing therein. In fact, the forcing is sufficiently powerful to produce a discontinuity in the vortex-spanwise shear at the boundary-layer/critical-layer edges. Later, the Hall–Smith theory was extended to include cross-flow effects [27] and also examined close to the input station [1,2,21–23,39]. In the paper by Brown *et al.*, the authors were keen to see how the Hall–Smith interaction starts and they discovered that the wave-pressure amplitude possess an algebraic bifurcation from a zero value to a non-zero value. A shorter streamwise length scale was subsequently identified in Smith *et al.* [3] over which previously neglected streamwise variations of the wave pressure come into play, allowing the aforesaid discontinuity or bifurcation to be removed/smoothed out. As a result a number of alternative transition paths were identified on this shorter length scale. Further, a common feature in both of the last-two mentioned papers is the existence of a thin buffer layer, embedding the even thinner critical layer and within which the main viscous-inviscid mean motion is effectively localised.

Here we are interested in adapting the interactive flow structure derived in the incompressible regime by Smith *et al.* [3] to accommodate increased compressibility; this is firstly for $O(1)$ values of the Mach number and secondly in the hypersonic limit of large Mach number. Regarding the latter, we observe that two main types of neutral inviscid mode are known to exist in highly compressible flows, these being the acoustic modes [6,9,10,30] and the vorticity mode [9]. Whereas the acoustic modes occupy the entire boundary layer, the vorticity mode is in effect confined to a relatively thin layer which resides in the outer reaches of the boundary layer, is sometimes termed the vorticity layer, and embeds the even-thinner critical layer. The vorticity layer itself emerges as a result of the relatively high basic-flow temperature having to adjust to that of the much cooler free stream, and hence is often referred to alternatively as the 'temperature-adjustment' layer. Of the two types of mode above, the vorticity mode has a considerably higher growth rate and so is naturally of most concern to us. Part of our aim indeed in this study is to find out the nonlinear counterpart of the vorticity mode when it becomes involved in a wave/vortex interaction of the type described above in hypersonic flow. This should allow eventually a firm link to be established with the experimental results by Holden [5] of transition emanating from the outer reaches of the boundary layer at high Mach numbers. We mention in passing that several reviews have been published recently (e.g. Cheng [31,32], Treanor [33]), which offer useful background knowledge on high-Mach-number modes as well as many other topics of interest in hypersonic flows; again, the present theory applies to wakes (see [34]) and other flows as well as boundary layers.

In Section 2 the formulation of the interactive flow problem is given, including a derivation of the important scalings in the wave/vortex interaction. The interaction problem for a compressible boundary layer at order-one Mach number is presented in Section 3, resulting in a nonlinear integro-differential equation governing the amplitude response of the wave. Certain comparisons are made here with the incompressible counterpart in [3], especially with regard to the next stage in the flow development downstream (see also independent work by Leib and Lee [35]). In fact, the broad form of the equation is identical (albeit with different coefficients of course) with the incompressible case and consequently the downstream options or different transition paths are invariant. These are downstream saturation, finite-distance singularity, pure-vortex motion downstream and nonlinear periodic motion, and are explained in detail at the end of Section 3. The flow solution is then examined for the regime of high Mach number in Section 4; the results of Smith and Brown [9] and others would tend to suggest overall that high-Mach-number predictions in practice can work well in the Mach-number range beyond about 3 or 4, in contrast with a subsequent study by Blackaby, Cowley and Hall [12], although really the range is unknown in advance. It is shown in Section 4 that for the nonlinear extension of the vorticity mode the interaction is essentially confined to the temperature-adjustment layer, due to the rapid decay of the wave part outside this layer. Hence the previous importance of the linear Stokes or wall-layer contribution to the evolution of the wave amplitude now becomes negligible. The special case of a model fluid (that is, a Chapman fluid with unit Prandtl number) is considered later in this section, and interesting results are obtained. In particular, it is found that for sufficiently high wall temperatures no periodic solutions are possible and that the wave angle of propagation has to lie between 45° and 90°, whereas for lower temperatures only periodic solutions are possible with the wave angle between 0° and 45°. Further developments of the wave/vortex interaction, including speculation on the first breakdown of the analysis due to further increases in the Mach number, along with final comments are given in Section 5.

In the following work, the velocities $u_\infty(u, v, w)$, the pressure $\rho_\infty a^2\gamma^{-1}p$, the Cartesian coordinates $l_\infty(x, y, z)$, the time $l_\infty u_\infty^{-1}t$, the density $\rho_\infty\rho$, the viscosity $\mu_\infty\mu$ and the temperature $a^2(\gamma-1)^{-1}c_p^{-1}T$ are scaled in the form

$$[u, v, w, p, x, y, z, t, \rho, \mu, T] = [\bar{u}, \epsilon^6\bar{v}, \epsilon^6\bar{w}, \bar{p}, \bar{x}, \epsilon^6\bar{y}, \epsilon^6\bar{z}, \bar{t}, \bar{\rho}, \bar{\mu}, \bar{T}]. \tag{1.1a–k}$$

Here $u_\infty, \rho_\infty, \mu_\infty$ represent the free-stream values of the streamwise velocity, the density and the viscosity respectively, l_∞ denotes the typical global length scale (e.g. the distance from the leading edge of the flat plate or airfoil, or the chord length), $\gamma = c_p/c_v$ is the ratio of the specific heat at constant pressure to that at constant density, and a is the speed of sound. In (1.1*a*–*k*) above, $\epsilon = Re^{-1/12}$ is small, given that the global Reynolds number $Re = u_\infty l_\infty \rho_\infty \mu_\infty^{-1}$ is taken as a large parameter, and the Mach number $M_\infty = u_\infty a^{-1}$ is $O(1)$, at least initially. See Section 4 for extensions to the hypersonic range.

Substituting (1.1a–k) into the compressible Navier–Stokes equations gives the governing equations of motion

$$\bar{\rho}_{\bar{t}} + (\bar{\rho}\bar{u})_{\bar{x}} + (\bar{\rho}\bar{v})_{\bar{y}} + (\bar{\rho}\bar{w})_{\bar{z}} = 0, \tag{1.2a}$$

$$\begin{aligned}\bar{\rho}\bar{u}_{\bar{t}} + \bar{\rho}(\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \bar{w}\bar{u}_{\bar{z}}) &= -\bar{p}_{\bar{x}}/\gamma M_\infty^2 + \epsilon^{12}[(2\bar{\mu}\bar{u}_{\bar{x}})_{\bar{x}} + (\bar{\mu}\bar{v}_{\bar{x}})_{\bar{y}} + \\ &\quad +(\bar{\mu}\bar{w}_{\bar{x}})_{\bar{z}}] + (\bar{\mu}\bar{u}_{\bar{y}})_{\bar{y}} + (\bar{\mu}\bar{u}_{\bar{z}})_{\bar{z}},\end{aligned} \tag{1.2b}$$

$$\begin{aligned}\bar{\rho}\bar{v}_{\bar{t}} + \bar{\rho}(\bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \bar{w}\bar{v}_{\bar{z}}) &= -\epsilon^{-12}\bar{p}_{\bar{y}}/\gamma M_\infty^2 + \epsilon^{12}[(\bar{\mu}\bar{v}_{\bar{x}})_{\bar{x}}] + \\ &\quad +(\bar{\mu}\bar{u}_{\bar{y}})_{\bar{x}} + (2\bar{\mu}\bar{v}_{\bar{y}})_{\bar{y}} + [\bar{\mu}(\bar{w}_{\bar{y}} + \bar{v}_{\bar{z}})]_{\bar{z}},\end{aligned} \tag{1.2c}$$

$$\begin{aligned}\bar{\rho}\bar{w}_{\bar{t}} + \bar{\rho}(\bar{u}\bar{w}_{\bar{x}} + \bar{v}\bar{w}_{\bar{y}} + \bar{w}\bar{w}_{\bar{z}}) &= -\epsilon^{-12}\bar{p}_{\bar{z}}/\gamma M_\infty^2 + \epsilon^{12}[(\bar{\mu}\bar{w}_{\bar{x}})_{\bar{x}}] + (\bar{\mu}\bar{u}_{\bar{z}})_{\bar{x}} + \\ &\quad +[\bar{\mu}(\bar{w}_{\bar{y}} + \bar{v}_{\bar{z}})]_{\bar{y}} + (2\bar{\mu}\bar{w}_{\bar{z}})_{\bar{z}},\end{aligned} \tag{1.2d}$$

$$\bar{\rho}\bar{T} = \bar{p}, \qquad \bar{\mu} = C\bar{T}, \tag{1.2e,f}$$

$$\begin{aligned}\bar{\rho}\bar{T}_{\bar{t}} + \bar{\rho}(\bar{u}\bar{T}_{\bar{x}} + \bar{v}\bar{T}_{\bar{y}} + \bar{w}\bar{T}_{\bar{z}}) &= \sigma^{-1}[\epsilon^{12}(\bar{\mu}\bar{T}_{\bar{x}})_{\bar{x}} + (\bar{\mu}\bar{T}_{\bar{y}})_{\bar{y}} + (\bar{\mu}\bar{T}_{\bar{z}})_{\bar{z}}] + \\ &\quad +(\gamma-1)M_\infty^2(\bar{p}_{\bar{t}} + \bar{u}\bar{p}_{\bar{x}} + \bar{v}\bar{p}_{\bar{y}} + \bar{w}\bar{p}_{\bar{z}}) + \\ &\quad +(\gamma-1)M_\infty^2\bar{\mu}(\bar{u}_{\bar{y}}^2 + \bar{u}_{\bar{z}}^2).\end{aligned} \tag{1.2g}$$

These represent in turn the equations of continuity, momentum in the three Cartesian directions, state, Chapman's viscosity–temperature law, and energy, and they are analysed in detail in the subsequent sections. Finally here, we note that C is the Chapman constant while σ is the Prandtl number.

2. Formulation of the Problem

It is well known from classical linear instability theory that small three-dimensional inviscid inflectional disturbances (say of order h relative to the basic-flow size) in compressible flows are governed by the generalised linear Rayleigh equation for wave pressure, namely

$$\frac{\partial^2 p_w}{\partial \bar{y}^2} + \frac{\partial^2 p_w}{\partial \bar{z}^2} - 2\frac{\mathrm{d}\bar{M}/\mathrm{d}\bar{y}}{\bar{M}}\frac{\partial p_w}{\partial \bar{y}} - \alpha^2(1-\bar{M}^2)p_w = 0 \tag{2.1}$$

[6,30] for a two-dimensional input flow, subject to the boundary conditions

$$\frac{\partial p_w}{\partial \bar{y}}(0,\bar{z}) = p_w(\infty,\bar{z}) = 0. \tag{2.2a,b}$$

The scaled wave pressure here is written as '$p_w(\bar{y},\bar{z})E+$ complex conjugate', where

$$E \equiv \exp[i(\alpha(\bar{x}_0)X - \Omega\bar{t}_1)]. \tag{2.3}$$

The fast temporal coordinate is $\bar{t}_1 \equiv \epsilon^{-6}\bar{t}$ whereas its spatial streamwise counterpart X is defined by

$$\alpha(\bar{x}_0)X = \epsilon^{-6}\int \alpha(\bar{x})\,\mathrm{d}\bar{x}. \tag{2.4}$$

Here for a neutral disturbance Ω is a prescribed real frequency while $\alpha(\bar{x})$ (also real) defines the variable streamwise wavenumber, which has to be determined from the analysis. Also, $\bar{x} = \bar{x}_0$ denotes the upstream neutral starting point of the wave/vortex interaction.

The main effect so far on the wave due to the basic flow is reflected by the term

$$\bar{M} \equiv (\bar{u} - c)M_\infty/\bar{T}^{1/2} \tag{2.5}$$

in (2.1) above. Here $\bar{u}, \bar{T}$ are the velocity and temperature profiles of the basic flow, respectively, and $c(\bar{x})$ is the effective wavespeed, satisfying

$$c(\bar{x}) = \Omega/\alpha(\bar{x}), \tag{2.6}$$

that would determine the temporal instability of the perturbation if linear. Since we are examining near-neutral disturbances, the quantity c is real to leading order, from which it can be deduced that a regular solution in wave pressure is only allowable if we impose the generalised inflection-point condition

$$\frac{\mathrm{d}^2\bar{M}}{\mathrm{d}\bar{y}^2} = 0 \quad \text{at} \quad \bar{y} = f(\bar{x}, \bar{z}). \tag{2.7}$$

Here f denotes the critical-level surface which is the solution of $\bar{u} = c(\bar{x})$ [36,37].

The algebraic response of the three-dimensional wave near the point of inflection is given by

$$[u_w, v_w, w_w, p_w] \sim h[(\bar{y} - f)^{-1}, 1(\bar{y} - f)^{-1}, 1] \tag{2.8a–d}$$

as $\bar{y} \to f^\pm$, where (u_w, v_w, w_w) denotes the wave velocity. The singularities here are damped out in a thin viscous critical layer of relative thickness $O(\epsilon^2)$ which embeds the inflectional surface $\bar{y} = f(\bar{x}, \bar{y})$.

It follows that wave–wave interaction is considerably larger in the critical layer, compared with anywhere else within the boundary layer, as a result of the amplified disturbance speed there, i.e. in (2.8a,c), and consequently the wave forcing on the vortex is essentially confined to this layer and just outside it. In particular, from the spanwise momentum balances, the spanwise velocity component of the vortex w_v satisfies

$$\frac{\mathrm{d}^2 w_v}{\mathrm{d}Y^2} = h^2\epsilon^{-6}\left\langle \bar{u}_w\frac{\partial\bar{w}_w}{\partial X} + \bar{v}_w\frac{\partial\bar{w}_w}{\partial Y} + \bar{w}_w\frac{\partial\bar{w}_w}{\partial\bar{z}}\right\rangle \tag{2.9}$$

where $\langle\ \rangle$ alludes to the vortex component of the enclosed expression, Y is the local critical-layer coordinate as defined later and $(\bar{u}_w, \bar{v}_w, \bar{w}_w)$ is the scaled representation of the wave velocity. To account for the dynamics in the critical layer as reflected in (2.9), the spanwise velocity component of the vortex must be discontinuous in its first derivative across $\bar{y} = f$, satisfying the condition

$$\left[\frac{\mathrm{d}w_v}{\mathrm{d}\bar{y}}\right]_{f_-}^{f^+} = h^2\epsilon^{-8}J(\bar{x}, \bar{z}). \tag{2.10}$$

Here J is a measure of the scaled wave-wave forcing, whose value is shown later to have the form

$$J(\bar{x}, \bar{z}) = K\frac{\partial}{\partial\bar{z}}\left(\left|\frac{\partial\bar{p}_w}{\partial\bar{z}}\right|\right)^2,$$

with $\bar{p}_w(\bar{x}, \bar{z})$ being the critical-layer wave pressure and K a known constant. Through the continuity balances, we deduce that the streamwise component of vortex velocity has magnitude $O(h^2\epsilon^{-14})$, so that significant interaction follows when the input amplitude is

$$h \sim \epsilon^7. \tag{2.11}$$

It is interesting to note that the order of magnitude of the wave amplitude remains unaffected as $M_\infty \to 0$, *i.e.* as the incompressible case is recovered [3]. On the other hand for $M_\infty \gg 1$ we shall find that the wave amplitude diminishes in order to retain nonlinear interaction: see Section 4 below.

As stated previously, we are interested in studying the wave/vortex interaction close to the input (neutral) station. The specific streamwise scale we seek emerges from the balance of nonparallelism and linear streamwise variations, i.e.

$$|(\bar{x} - \bar{x}_0)\bar{u}_1|\epsilon^{-6}\left|\frac{\partial}{\partial X}\right| \sim |\bar{u}_0|\left|\frac{\partial}{\partial \bar{x}}\right|,$$

where $\bar{u}_0 + (\bar{x} - \bar{x}_0)\bar{u}_1 + \cdots$ is the mean-flow profile expressed as a Taylor series about the starting point $\bar{x} = \bar{x}_0$. Hence, the required scaling is

$$\bar{x} - \bar{x}_0 = \epsilon^3 x_1, \tag{2.12}$$

say, with x_1 being $O(1)$.

As with the theories for the incompressible range in [3] and [2], the convective forces dominate over the viscous forces as regards the vortex motion nearly everywhere in the boundary layer (thus rendering the vortex inviscid there). However these forces counterbalance in a pair of thin so-called buffer layers (each having relative thickness $O(\epsilon^{3/2})$ for a viscous-inviscid balance) which are separated by the thinner critical layer, and it is within these regions that the wave/vortex interaction is effectively concentrated. A full diagram clarifying the details of the multi-layered structure is given in Figure 1.

3. Compressible Interactions in the Sub- or Supersonic Ranges

3.1. The Core Regions

These regions, where $\bar{y} = O(1)$, constitute the majority of the boundary layer (see Figure 1), and contain important linear and nonparallel forces that contribute to the alteration of the wave-amplitude downstream. The vortex motion turns out to be negligible here, only being a significant influence in the critical layer and the buffer layers. The following expansions hold, in view of the argument in Section 2 and [3],

$$\bar{u} = \bar{U}_0(\bar{y}) + \epsilon^3 x_1 \bar{U}_1(\bar{y}, \bar{z}) + \cdots + \epsilon^7 u^{(0)} E + \epsilon^{10} u^{(1)} E + \cdots, \tag{3.1a}$$

$$\bar{v} = \bar{V}_0(\bar{y}, \bar{z}) + \cdots + \epsilon v^{(0)} E + \epsilon^4 v^{(1)} E + \cdots, \tag{3.1b}$$

$$\bar{w} = \epsilon^3 x_1 \bar{W}_1(\bar{y}, \bar{z}) + \cdots + \epsilon w^{(0)} E + \epsilon^4 w^{(1)} E + \cdots, \tag{3.1c}$$

$$\bar{p} = \bar{p}_{00} + \epsilon^3 x_1 \bar{p}_0 + \cdots + \epsilon^7 p^{(0)} E + \epsilon^{10} p^{(1)} E + \cdots, \tag{3.1d}$$

$$\bar{\rho} = \rho_0(\bar{y}) + \epsilon^3 x_1 \bar{\rho}_1(\bar{y}, \bar{z}) + \cdots + \epsilon^7 \rho^{(0)} E + \epsilon^{10} \rho^{(1)} E + \cdots, \tag{3.1e}$$

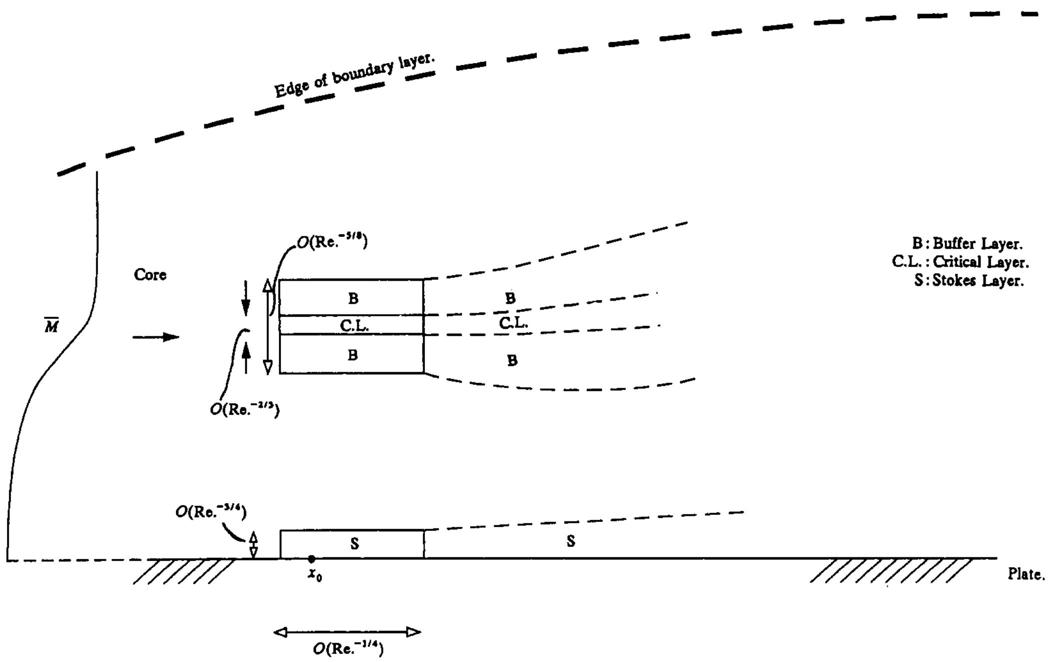

Figure 1. Sketch of the oncoming generalised mean-flow profile $\bar{M}(\propto (\bar{U}_0 - c_0)\bar{T}_0^{-1/2})$ and the subsequent wave/vortex-interaction flow structure for $M_\infty = O(1)$.

$$\bar{\mu} = \bar{\mu}_0(\bar{y}) + \epsilon^3 x_1 \bar{\mu}_1(\bar{y}, \bar{z}) + \cdots + \epsilon^7 \mu^{(0)} E + \epsilon^{10} \mu^{(1)} E + \cdots, \tag{3.1f}$$

$$\bar{T} = \bar{T}_0(\bar{y}) + \epsilon^3 x_1 \bar{T}_1(\bar{y}, \bar{z}) + \cdots + \epsilon^7 T^{(0)} E + \epsilon^{10} T^{(1)} E + \cdots, \tag{3.1g}$$

including nonparallel terms proportional to x_1. Here $\bar{p}_{00}$ and $\bar{p}_0$ denote the basic-flow pressure and its gradient evaluated locally, in turn, with the former being prescribed the value 1 in the ensuing work. Elsewhere, the contributions from the oncoming basic flow and its local gradient are given the subscripts 0 and 1, respectively. Also, the terms multiplying E (dependent on $x_1, \bar{y}, \bar{z}$) denote wave contributions with successive orders being given the superscript (0), (1), etc. Strictly we should include the complex conjugates of the wave terms in (3.1a–g), but these are omitted for the sake of brevity without affecting the following analysis.

Substituting (3.1a–g) into the compressible Equations (1.2a–g) determines the system of equations for each component of the flow, i.e. basic flow or wave. First, the basic flow satisfies

$$\bar{\rho}_1 \bar{U}_0 + \bar{\rho}_0 \bar{U}_1 + (\bar{\rho}_0 \bar{V}_0)_{\bar{y}} = 0, \tag{3.2a}$$

$$\bar{\rho}_0(\bar{U}_0 \bar{U}_1 + \bar{V}_0 \bar{U}_{0\bar{y}}) = -\bar{p}_0/\gamma M_\infty^2 + (\bar{\mu}_0 \bar{U}_{0\bar{y}})_{\bar{y}}, \tag{3.2b}$$

$$1 = \bar{\rho}_0 \bar{T}_0, \quad \bar{p}_0 = \bar{\rho}_0 \bar{T}_1 + \bar{\rho}_1 \bar{T}_0, \quad \bar{\mu}_0 = C\bar{T}_0, \tag{3.2c,d,e}$$

$$\bar{\rho}(\bar{U}_0 \bar{T}_1 + \bar{V}_0 \bar{T}_{0\bar{y}}) = \frac{(\gamma - 1)}{\gamma} \bar{p}_0 \bar{U}_0 + \sigma^{-1}(\bar{\mu}_0 \bar{T}_{0\bar{y}})_{\bar{y}} + (\gamma - 1) M_\infty^2 \bar{\mu}_0 \bar{U}_{0\bar{y}}^2. \tag{3.2f}$$

These equations may be solved to reveal that

$$\bar{V}_0 = \frac{\bar{U}_0}{\bar{\rho}_0} \left\{ \int_{\bar{a}_0}^{\bar{y}} \left(\frac{[\bar{p}_0/\gamma M_\infty^2 - (\bar{\mu}_0 \bar{U}_{0\bar{y}_1})_{\bar{y}_1}]}{\bar{U}_0^2} - \bar{\rho}_1 \right) \mathrm{d}\bar{y}_1 + \frac{\rho_0 V_0}{c_0} \right\}, \tag{3.3a}$$

$$\bar{U}_1 = -\frac{[\bar{p}_0/\gamma M_\infty^2 - (\bar{\mu}_0 \bar{U}_{0\bar{y}})_{\bar{y}}]}{\bar{\rho}_0 \bar{U}_0} + \\ + \frac{\bar{U}_{0\bar{y}}}{\bar{\rho}_0} \left\{ \int_{\bar{a}_0}^{\bar{y}} \left(\frac{[\bar{p}_0/\gamma M_\infty^2 - (\bar{\mu}_0 \bar{U}_{0\bar{y}_1})_{\bar{y}_1}]}{\bar{U}_0^2} - \bar{\rho}_1 \right) \mathrm{d}\bar{y}_1 + \frac{\rho_0 V_0}{c_0} \right\}, \tag{3.3b}$$

$$\bar{\rho}_1 = \bar{\rho}_{0\bar{y}} \left\{ \int_{\bar{a}_0}^{\bar{y}} \left(\frac{\bar{p}_0}{\gamma M_\infty^2} \left[\left(\frac{\bar{\rho}_0^2}{\bar{\rho}_{0\bar{y}_1}} \right)_{\bar{y}_1} \bar{U}_0^2 M_\infty^2 - 1 \right] + \right. \right. \\ + \frac{1}{\bar{U}_0^2 \bar{\rho}_0} (\bar{\mu}_0 \bar{U}_{0\bar{y}_1})_{\bar{y}_1} + \frac{1}{\sigma \bar{\rho}_0} \frac{\mathrm{d}}{\mathrm{d}\bar{y}_1} \left[\frac{\bar{\rho}_0^2 (\bar{\mu}_0 \bar{\rho}_{0\bar{y}_1}/\bar{\rho}_0^2)_{\bar{y}_1}}{\bar{U}_0 \bar{\rho}_{0\bar{y}_1}} \right] - \\ \left. \left. - \frac{1}{\bar{\rho}_0} (\gamma - 1) M_\infty^2 \frac{\mathrm{d}}{\mathrm{d}\bar{y}_1} \left[\frac{\bar{\mu}_0 \bar{U}_{0\bar{y}_1}^2 \bar{\rho}_0^2}{\bar{U}_0 \bar{\rho}_{0\bar{y}_1}} \right] \right) \mathrm{d}\bar{y}_1 + \frac{\rho_{10}}{\rho_1} \right\}. \tag{3.3c}$$

Here we have used the expansions

$$(\alpha(\bar{x}), c(\bar{x})) = (\alpha_0, c_0) + \epsilon^3 x_1 (\alpha_2, c_2) + \cdots, \tag{3.4}$$

for the streamwise wavenumber and wavespeed near $\bar{x} = \bar{x}_0$. The frequency, defined by $\Omega = \alpha c$, is a constant prescribed value in our problem, which therefore implies the constraint

$$\alpha_0 c_2 + \alpha_2 c_0 = 0. \tag{3.5}$$

The inflectional plane $f(\bar{x}, \bar{z})$ has the localised form

$$f(\bar{x}, \bar{z}) = \bar{a}_0 + \epsilon^3 x_1 \bar{a}_2 + \cdots, \tag{3.6}$$

where $\bar{a}_0$ and $\bar{a}_2$ are both constants, owing to the spanwise independence of $\bar{U}_0, \bar{U}_1$ and c. Lastly, the terms $\rho_0, \rho_1, \rho_{10}, V_0$ simply represent $\bar{\rho}_0, \bar{\rho}_{0\bar{y}}, \bar{\rho}_1, \bar{V}_0$, in turn, evaluated at $\bar{y} = \bar{a}_0$. We observe that the values of V_0, ρ_{10} are determined from the boundary conditions at the wall, i.e. $\bar{U}_1 = \bar{V}_0 = 0, \bar{\rho}_1 = \rho_{1w}$ at $\bar{y} = 0$, where ρ_{1w} is the specified basic-flow density gradient at the wall.

Next, we focus on the leading-order wave system which reduces to the generalised three-dimensional Rayleigh equation

$$p_{\bar{y}\bar{y}}^{(0)} + p_{\bar{z}\bar{z}}^{(0)} - 2\frac{\bar{M}_{\bar{y}}}{\bar{M}} p_{\bar{y}}^{(0)} - \alpha_0^2 (1 - \bar{M}^2) p^{(0)} = 0, \tag{3.7}$$

subject to

$$p^{(0)} \to 0 \quad \text{as} \quad \bar{y} \to \infty, \qquad p_{\bar{y}}^{(0)} = 0 \quad \text{at} \quad \bar{y} = 0. \tag{3.8a,b}$$

Here

$$\bar{M} \equiv (\bar{U}_0 - c_0) M_\infty / \bar{T}_0^{1/2}, \tag{3.9}$$

which in effect represents the basic-flow forcing on the wave, is independent of $\bar{z}$. Thus we are able to separate variables in $p^{(0)}$ and in particular we may write

$$p^{(0)} = r(x_1) P_0(\bar{y}) \cos \beta_0 \bar{z}, \tag{3.10}$$

for a pair of equal waves $[\exp(\pm i\beta_0 \bar{z})]$ obliquely inclined to the free-stream direction, with β_0 constant. Here r is the amplitude function of the wave, while P_0 satisfies

$$P_{0\bar{y}\bar{y}} - 2\frac{\bar{M}_{\bar{y}}}{\bar{M}} P_{0\bar{y}} - [\gamma_0^2 - \alpha_0^2 \bar{M}^2] P_0 = 0, \tag{3.11}$$

with

$$P_0(\infty) = 0, \qquad P_{0\bar{y}}(0) = 0, \tag{3.12a,b}$$

from (3.7), (3.8a,b), where $\gamma_0^2 = (\alpha_0^2 + \beta_0^2)$ is a constant. This system determines in principle the eigensolution for P_0 and the corresponding characteristic equation that constrains the parameters α_0, β_0 and c_0. It is convenient to make the normalisation $P_0(\bar{a}_0) = 1$, which we are free to do, owing to the homogeneous nature of (3.11), (3.12a,b).

To fully understand the dynamics of the buffer layer (and, to some extent, the critical layer) we need to know the behaviour of the flow solution as $\bar{y} = \bar{a}_0$ is approached. The important results are now summarised. As $\bar{y} \to \bar{a}_0^{\pm}$,

$$\bar{U}_0 = c_0 + \sum_{n=1}^{\infty} \frac{b_n(\bar{y} - \bar{a}_0)^n}{n!}, \qquad \bar{U}_1 = \sum_{n=0}^{\infty} \frac{d_n(\bar{y} - \bar{a}_0)^n}{n!}, \tag{3.13a,b}$$

$$P_0 = 1 + \sum_{n=1}^{\infty} \frac{q_n(\bar{y} - \bar{a}_0)^n}{n!}. \tag{3.13c}$$

Also

$$\{\bar{\rho}_0, \bar{\rho}_1\} = \sum_{n=0}^{\infty} \{\rho_n, \rho_{1n}\} \frac{(\bar{y} - \bar{a}_0)^n}{n!} \tag{3.13d,e}$$

as $\bar{y} \to \bar{a}_0^{\pm}$ and, likewise, similar expressions hold for $\bar{\mu}_0, \bar{\mu}_1, \bar{T}_0, \bar{T}_1$. Here the coefficients satisfy

$$d_0 = \frac{-1}{\rho_0 c_0}\left[\frac{p_0}{\gamma M_\infty^2} + b_1 \rho_0 V_0 + \frac{2Cb_1\rho_1}{\rho_0^2}\right], \tag{3.14a}$$

$$d_1 = \frac{1}{\rho_0 c_0}\left[\frac{\rho_1}{\rho_0}\left(\frac{p_0}{\gamma M_\infty^2}\right) + b_1 c_0 \rho_{10} + \frac{C}{\rho_0}\left(b_3 - \frac{b_1\rho_2}{\rho_0}\right) + \frac{2Cb_1\rho_1}{\rho_0^2}\left(\frac{3\rho_1}{\rho_0} + \frac{\rho_0^2 V_0}{C}\right)\right], \tag{3.14b}$$

$$\begin{aligned} d_2 = \frac{1}{\rho_0 c_0}\Bigg[&\left(\frac{\rho_2}{\rho_0} + \frac{\rho_1}{\rho_0}\left(\frac{b_1}{c_0} - \frac{2\rho_1}{\rho_0}\right)\right)\left(\frac{p_0}{\gamma M_\infty^2}\right) + \frac{C}{\rho_0}\left(b_4 - \frac{b_1\rho_3}{\rho_0}\right) + \\ &+ b_1 c_0\left(\rho_{11} - \frac{4\rho_1\rho_{10}}{\rho_0}\right) - \frac{C}{\rho_0}\left(\left(\frac{b_1}{c_0} + \frac{2\rho_1}{\rho_0}\right) + \frac{\rho_0^2 V_0}{C}\right)\left(b_3 - \frac{b_1\rho_2}{\rho_0}\right) - \\ &- \frac{3C\rho_1 b_3}{\rho_0^2} - \frac{Cb_1\rho_1}{\rho_0^2}\left(\frac{2\rho_1}{\rho_0}\left(\frac{b_1}{c_0} + \frac{12\rho_1}{\rho_0}\right) - \frac{11\rho_2}{\rho_0} + \frac{4\rho_0\rho_1 V_0}{C}\right)\Bigg], \end{aligned} \tag{3.14c}$$

from (3.3b), and

$$q_1 = 0, \qquad q_2 = -\gamma_0^2, \tag{3.15a,b}$$

$$q_4 = -\gamma_0^2\left[\frac{6\rho_2}{\rho_0} - \frac{9\rho_1^2}{\rho_0^2} + 3\gamma_0^2 + \frac{4b_3}{b_1}\right] - 6\alpha_0^2 b_1^2 \rho_0 M_\infty^2, \tag{3.15c}$$

from (3.11). It is important to note that the generalised inflection-point condition $\mathrm{d}^2M/\mathrm{d}\bar{y}^2 = 0$ at $\bar{y} = 0$ implies that

$$b_2 + b_1\rho_1/\rho_0 = 0 \tag{3.16}$$

to leading order, and this has been used extensively in the above derivations. Again, the local third derivative of the wave pressure q_3 cannot be determined by local asymptotic analysis but instead has to be found by solving the entire Rayleigh system (3.11), (3.12a,b).

It is worthwhile here writing down the local series expansion for $\bar{M}$, about $\bar{y} = \bar{a}_0$. It is

$$\bar{M} = M_1(\bar{y} - \bar{a}_0) + \sum_{n=3}^{\infty} \frac{M_n(\bar{y} - \bar{a}_0)^n}{n!} \tag{3.17}$$

as $\bar{y} \to \bar{a}_0^{\pm}$, where

$$M_1 = b_1\rho_0^{1/2}M_\infty, \tag{3.18a}$$

$$M_3 = \left[b_3 + \frac{3b_1\rho_2}{2\rho_0} - \frac{9b_1\rho_1^2}{4\rho_0^2}\right]\rho_0^{1/2}M_\infty, \tag{3.18b}$$

$$M_4 = \left[b_4 + \frac{2b_3\rho_1}{\rho_0} + b_1\left(\frac{2\rho_3}{\rho_0} - \frac{6\rho_1\rho_2}{\rho_0^2} + \frac{3\rho_1^3}{\rho_0^3}\right)\right]\rho_0^{1/2}M_\infty. \tag{3.18c}$$

We observe that the absence of any coefficient multiplying $(\bar{y}-\bar{a}_0)^2$ in the above expansion is simply due to the inflection-point condition.

Finally in the core, we need to examine the second-order wave system, which is governed by a small-perturbation form of (3.11) above, that is

$$\begin{aligned}
&p_{\bar{y}\bar{y}}^{(1)} + p_{\bar{z}\bar{z}}^{(1)} - \frac{2\bar{M}_{\bar{y}}}{\bar{M}}p_{\bar{y}}^{(1)} - \alpha_0^2(1-\bar{M}^2)p^{(1)} \\
&= \frac{2x_1}{\bar{M}}\Bigg[\left(\alpha_0\alpha_2\bar{M} - (\alpha_0^2\bar{M}_1 + \alpha_0\alpha_2\bar{M})\bar{M}^2\right)p^{(0)} + \\
&\quad + \left(\bar{M}_{1\bar{y}} - \frac{\bar{M}_1\bar{M}_{\bar{y}}}{\bar{M}}\right)p_{\bar{y}}^{(0)} + \left(\bar{M}_{1\bar{z}} - \frac{\bar{M}_1\bar{M}_{\bar{z}}}{\bar{M}}\right)p_{\bar{z}}^{(0)}\Bigg] - \\
&\quad - \frac{2ic_0}{\alpha_0\bar{M}}\frac{\partial}{\partial x_1}\Bigg[\left((\bar{\rho}_0^{1/2})_{\bar{y}} - \frac{\bar{\rho}_0^{1/2}\bar{M}_{\bar{y}}}{\bar{M}}\right)M_\infty p_{\bar{y}}^{(0)} + \\
&\quad + \frac{\alpha_0^2}{c_0}\left(\bar{M}(1-\bar{M}^2) - c_0\bar{\rho}_0^{1/2}M_\infty\bar{M}^2\right)p^{(0)}\Bigg],
\end{aligned} \tag{3.19}$$

from (1.2a–g), where

$$\bar{M}_1 = \left[\frac{(\bar{U}_1 - c_2)}{(\bar{U}_0 - c_0)} + \frac{1}{2}\left(\frac{\bar{\rho}_1}{\bar{\rho}_0} - p_0\right)\right]\bar{M}. \tag{3.20}$$

The boundary conditions to be satisfied by $p^{(1)}$ are

$$p^{(1)} \to 0 \quad \text{as} \quad \bar{y} \to \infty, \tag{3.21a}$$

$$p^{(1)} \text{ matches with the Stokes' layer solution at the wall.} \tag{3.21b}$$

To analyse the pressure system further, we write

$$p^{(1)} = P_0(\bar{y})Q_1(x_1, \bar{y})\cos\beta_0\bar{z} \tag{3.22}$$

where, for this spanwise representation to be valid, it is necessary to impose the condition $\bar{M}_{1\bar{z}} \equiv 0$ (and hence $\bar{\rho}_{1\bar{z}} \equiv 0$). After substituting (3/22) into (3.19) and equating coefficients of $\cos\beta_0\bar{z}$, we find that

$$P_0\frac{\partial^2 Q_1}{\partial \bar{y}^2} + 2\left(P_{0\bar{y}} - \frac{P_0\bar{M}_{\bar{y}}}{\bar{M}}\right)\frac{\partial Q_1}{\partial \bar{y}} = \frac{x_1 r(x_1)}{\bar{M}}G_1(\bar{y}) + \frac{ic_0 r'(x_1)}{\bar{M}}G_2(\bar{y}), \tag{3.23}$$

where the prime $'$ denotes differentiation with respect to x_1. Here

$$G_1 = 2\left[\left(\alpha_0\alpha_2\bar{M} - (\alpha_0^2\bar{M}_1 + \alpha_0\alpha_2\bar{M})\bar{M}^2\right)P_0 + \left(\bar{M}_{1\bar{y}} - \frac{\bar{M}_1\bar{M}_{\bar{y}}}{\bar{M}}\right)P_{0\bar{y}}\right], \tag{3.24a}$$

$$G_2 = \frac{2}{\alpha_0}\left[\left(\frac{\bar{\rho}_0^{-1/2}\bar{M}_{\bar{y}}}{\bar{M}} - (\bar{\rho}_0^{-1/2})_{\bar{y}}\right)M_\infty P_{0\bar{y}} - \frac{\alpha_0^2}{c_0}\left(\bar{M}(1-\bar{M}^2) - c_0\bar{\rho}_0^{-1/2}M_\infty\bar{M}^2\right)P_0\right] \tag{3.24b}$$

are measures of the nonparallel and streamwise-variation forces, in turn. The equation for Q_1 integrates to

$$\frac{\partial Q_1}{\partial \bar{y}}\frac{P_0^2}{\bar{M}^2} = \int_{0,\infty}^{\bar{y}} \frac{P_0}{\bar{M}^3}\left(x_1 r(x_1)G_1(\bar{y}_1) + ic_0 r'(x_1)G_2(\bar{y}_1)\right)\mathrm{d}\bar{y}_1 + + \frac{\hat{P}_w}{c_0^2\rho_w M_\infty^2}Q_w^{\mp} r(x_1), \tag{3.25}$$

for $\bar{y}$ less than or greater than $\bar{a}_0$ in turn, where $\hat{P}_w, \rho_w$ denote the constant values of P_0 and ρ_0 evaluated at $\bar{y} = 0$. Here $Q_w^+ = 0$ while Q_w^- is a complex constant determined through matching with the Stokes' layer solution as $\bar{y} \to 0^+$ (see Appendix A). Specifically we have

$$Q_w^- = \left[-\frac{\gamma_0^2}{\rho_w} + \frac{\alpha_0^2 c_0^2\rho_w\gamma M_\infty^2}{\sigma^{1/2}}\left(\hat{T}_w - \frac{1}{\rho_w}\frac{(\gamma-1)}{\gamma}\right)\right]\hat{P}_w\left(\frac{-i\alpha_0 c_0}{C}\right)^{-1/2}, \tag{3.26}$$

where $\hat{P}_w\hat{T}_w r(x_1)\cos\beta_0\bar{z}$ denotes the wave temperature at the wall, with $\hat{T}_w$ being constant-valued.

It can be deduced that as $\bar{y} \to \bar{a}_0^{\pm}$ the following asymptotic behaviour holds for $p^{(1)}$:

$$[p^{(1)}(x_1,\bar{y},\bar{z}) - p^{(1)}(x_1,\bar{a}_0,\bar{z})]\sec\beta_0\bar{z}$$
$$\sim \frac{-\gamma_0^2}{2}(\bar{y}-\bar{a}_0)^2 p^{(1)}(x_1,\bar{a}_0,\bar{z})\sec\beta_0\bar{z} +$$
$$+ M_1^2(\bar{y}-\bar{a}_0)\Big\{ x_1 r(x_1)\Big[\frac{-a_{-3}}{2} - \frac{a_{-2}}{2}(\bar{y}-\bar{a}_0)$$
$$+\frac{a_{-1}}{3}(\bar{y}-\bar{a}_0)^2\ln|\bar{y}-\bar{a}_0| + \frac{1}{3}G_a^{\pm}(\bar{y}-\bar{a}_0)^2\Big] +$$
$$+ ic_0 r'(x_1)\Big[\frac{-b_{-3}}{2} - \frac{b_{-2}}{2}(\bar{y}-\bar{a}_0) + \frac{b_{-1}}{3}(\bar{y}-\bar{a}_0)^2\ln|\bar{y}-\bar{a}_0| + \frac{1}{3}G_b^{\pm}(\bar{y}-\bar{a}_0)^2\Big] +$$
$$+ \frac{\tilde{P}_w}{3c_0^2\rho_w M_\infty^2} Q_w^{\pm} r(x_1)(\bar{y}-\bar{a}_0)^2\Big\}, \tag{3.27}$$

which reveals a jump in its third derivative there. Here

$$a_{-3} = 2q_2(c_2-d_0)\frac{\rho_0^{1/2}M_\infty}{M_1^3}, \tag{3.28a}$$

$$a_{-2} = \left(q_3(c_2-d_0) - \frac{2\alpha_0^2 b_1 c_2}{c_0}\right)\frac{\rho_0^{1/2}M_\infty}{M_1^3}, \tag{3.28b}$$

$$a_{-1} = q_2\Bigg((c_2-d_0)\left(\frac{M_3}{M_1} + \frac{\rho_1^2}{4\rho_0^2} - \frac{\rho_2}{2\rho_0}\right) + d_2 + \frac{d_1\rho_1}{\rho_0}$$
$$+b_1\left(\frac{\rho_{11}}{\rho_0} - \frac{\rho_1\rho_{10}}{\rho_0^2}\right)\Bigg)\frac{\rho_0^{1/2}M_\infty}{M_1^3}, \tag{3.28c}$$

and

$$b_{-3} = \left(\frac{2q_2}{\alpha_0}\right)\frac{\rho_0^{1/2}M_\infty}{M_1^3}, \tag{3.29a}$$

$$b_{-2} = \left(\frac{q_3}{\alpha_0} - \frac{2\alpha_0 b_1}{c_0}\right)\frac{\rho_0^{1/2}M_\infty}{M_1^3}, \tag{3.29b}$$

$$b_{-1} = \frac{q_2}{\alpha_0}\left(\frac{M_3}{M_1} + \frac{1}{4}\left(\frac{\rho_1^2}{\rho_0^2} - \frac{2\rho_2}{\rho_0}\right)\right)\frac{\rho_0^{1/2}M_\infty}{M_1^3}, \tag{3.29c}$$

where $a_n, b_n\ (n = -3,-2,\ldots)$ denote the coefficients of $(\bar{y}-\bar{a}_0)^n$ in the local Laurent expansions of $P_0G_1/\bar{M}^3, P_0G_2/\bar{M}^3$ respectively, *i.e.*

$$P_0G_1/\bar{M}^3 = \sum_{n=-3}^{\infty} a_n(\bar{y}-\bar{a}_0)^n, \qquad P_0G_2/\bar{M}^3 = \sum_{n=-3}^{\infty} b_n(\bar{y}-\bar{a}_0)^n. \tag{3.30a,b}$$

Lastly, the constant coefficients $G_{\bar{a}}^{\pm}, G_{\bar{b}}^{\pm}$ appearing in (3.27) have the differences

$$G_{\bar{a}}^{+} - G_{\bar{a}}^{-} = -\mathrm{FP}\int_0^\infty \frac{P_0}{\bar{M}^3} G_1(\bar{y})\,\mathrm{d}\bar{y} - \frac{a_{-3}}{2\bar{a}_0^2} + \frac{a_{-2}}{\bar{a}_0}, \tag{3.31a}$$

$$G_{\bar{b}}^{+} - G_{\bar{b}}^{-} = -\mathrm{FP}\int_0^\infty \frac{P_0}{\bar{M}^3} G_2(\bar{y})\,\mathrm{d}\bar{y} - \frac{b_{-3}}{2\bar{a}_0^2} + \frac{b_{-2}}{\bar{a}_0}, \tag{3.31b}$$

and these figure significantly in the later analysis. Here FP denotes the principal value or finite part.

3.2. The Buffer Layers

These relatively thin layers lie between the cores and the critical layer (see Figure 1). They are formally defined by

$$\bar{y} - f(\bar{x}, \bar{z}) = \epsilon^{3/2} Y_1 \tag{3.32}$$

where Y_1 is $O(1)$ and the critical-level surface f is defined in (3.6) above.

The flow variables expand as

$$\begin{aligned}\bar{u} = {} & c_0 + \epsilon^{3/2} b_1 Y_1 + \epsilon^3 (b_2 Y_1^2/2 + c_2 x_1) + \epsilon^{9/2}(b_3 Y_1^3/6 + (b_2\bar{a}_2 + d_1)x_1 Y_1) + \epsilon^6 u_4 + \\ & + \cdots + E\epsilon^{11/2}(\tilde{u}_0 + \epsilon^{3/2}\tilde{u}_1 + \epsilon^3 \tilde{u}_2 + \epsilon^{9/2}(\tilde{u}_l \ln\epsilon + \tilde{u}_3) + \cdots) + \cdots,\end{aligned} \tag{3.33a}$$

$$\begin{aligned}\bar{v} = {} & V_0 + \epsilon^{3/2}(\bar{a}_2 b_1 - c_2 - \rho_1 V_0/\rho_0 - c_0\rho_{10}/\rho_0) Y_1 + \epsilon^3 v_2 + \\ & + \cdots + E\epsilon(\tilde{v}_0 + \epsilon^{3/2}\tilde{v}_1 + \epsilon^3 v_2 + \cdots) + \cdots,\end{aligned} \tag{3.33b}$$

$$\bar{w} = \epsilon^{3/2} w_0 + E\epsilon^{-1/2}(\tilde{w}_0 + \epsilon^{3/2}\tilde{w}_1 + \epsilon^3 w_2 + \cdots) + \cdots, \tag{3.33c}$$

$$\begin{aligned}\bar{p} = {} & 1 + \epsilon^3 x_1 p_0 + \epsilon^6 x_1^2 p_1/2 + \\ & + E\epsilon^7(\tilde{p}_0 + \epsilon^{3/2}\tilde{p}_1 + \epsilon^3\tilde{p}_2 + \epsilon^{9/2}\tilde{p}_3 + \epsilon^6\tilde{p}_4 + \epsilon^{15/2}(\tilde{p}_l \ln\epsilon + \tilde{p}_5) + \cdots) + \cdots,\end{aligned} \tag{3.33d}$$

$$\begin{aligned}\bar{\rho} = {} & \rho_0 + \epsilon^{3/2}\rho_1 Y_1 + \epsilon^3(\rho_2 Y_1^2/2 + x_1(\rho_1\bar{a}_2 + \rho_{10})) + \\ & + \epsilon^{9/2}(\rho_3 Y_1^3/6 + (\rho_2\bar{a}_2 + \rho_{11})x_1 Y_1) + \epsilon^6 \rho_4 + \\ & + \cdots + E\epsilon^{11/2}(\tilde{\rho}_0 + \epsilon^{3/2}\tilde{\rho}_1 + \epsilon^3\tilde{\rho}_2 + \epsilon^{9/2}(\tilde{\rho}_l \ln\epsilon + \tilde{\rho}_3) + \cdots) + \cdots,\end{aligned} \tag{3.33e}$$

$$\begin{aligned}\bar{\mu} = {} & \mu_0 + \epsilon^{3/2}\mu_1 Y_1 + \epsilon^3(\mu_2 Y_1^2/2 + x_1(\mu_1\bar{a}_2 + \mu_{10})) + \\ & + \epsilon^{9/2}(\mu_3 Y_1^3/6 + (\mu_2\bar{a}_2 + \mu_{11})x_1 Y_1) + \epsilon^6 \mu_4 + \\ & + \cdots + E\epsilon^{11/2}(\tilde{\mu}_0 + \epsilon^{3/2}\tilde{\mu}_1 + \epsilon^3\tilde{\mu}_2 + \epsilon^{9/2}(\tilde{\mu}_l \ln\epsilon + \tilde{\mu}_3) + \cdots) + \cdots,\end{aligned} \tag{3.33f}$$

$$\begin{aligned}\bar{T} = {} & T_0 + \epsilon^{3/2} T_1 Y_1 + \epsilon^3(T_2 Y_1^2/2 + x_1(T_1\bar{a}_2 + T_{10})) + \\ & + \epsilon^{9/2}(T_3 Y_1^3/6 + (T_2\bar{a}_2 + T_{11})x_1 Y_1) + \epsilon^6 T_4 + \\ & + \cdots + E\epsilon^{11/2}(\tilde{T}_0 + \epsilon^{3/2}\tilde{T}_1 + \epsilon^3\tilde{T}_2 + \epsilon^{9/2}(\tilde{T}_l \ln\epsilon + \tilde{T}_3) + \cdots) + \cdots.\end{aligned} \tag{3.33g}$$

Here we note that the algebraic terms in (3.3a–g) stem from the Taylor-series representations of the basic-flow quantities at $\bar{y} = f$, which are defined in Section 3.1 above. The constant $\bar{a}_2$, defined in (3.6) above, denotes the critical-layer slope and is related to c_2 by

$$b_1\bar{a}_2 + d_0 = c_2. \tag{3.34}$$

The terms $u_4, v_2, w_0, \rho_4, \mu_4, T_4$ denote the mean-flow/vortex contributions to the flow, which are non-trivially affected by wave-wave forcing. This forcing is of an amplitude-squared type and emerges from the critical layer; in particular, it produces a discontinuity in vortex spanwise-shear across $\bar{y} = f$ as discussed in Section 2 above and shown in detail in Appendix B below. In addition to the above list of wave-affected vortex terms there exists a corresponding pressure term (of $O(\epsilon^{27/2})$ relative to the free-stream pressure value), but this is totally passive regarding the main vortex/wave interaction and need not be addressed here. Again the terms multiplying E denote the wave contributions.

We consider firstly the vortex motion, where spanwise-momentum balances yield the diffusion equation

$$\rho_0 c_0 \frac{\partial w_0}{\partial x_1} = \mu_0 \frac{\partial^2 w_0}{\partial Y_1^2} \tag{3.35}$$

for w_0. The boundary conditions on w_0 are

$$w_0 \to 0 \quad \text{as} \quad |Y_1| \to \infty \quad \text{and as} \quad x_1 \to -\infty \tag{3.36a}$$

and

$$\left[\frac{\partial w_0}{\partial Y_1}\right]_{Y_1=0^-}^{Y_1=0^+} = J_0(x_1, \bar{z}). \tag{3.36b}$$

The latter boundary condition stems from the critical-layer analysis of Appendix B, where we can also deduce that w_0 must be continuous across $\bar{y} = f$. Here

$$J_0 = \frac{1}{(\gamma M_\infty^2)^2} \frac{2\pi(2/3)^{2/3}(-2/3)!}{(\alpha_0 b_1)^{5/3}(C^2/\rho_0)^{2/3}} \frac{\partial}{\partial \bar{z}} \left(\left| \frac{\partial \tilde{p}_0}{\partial \bar{z}} \right|^2 \right), \tag{3.37}$$

with $(\tilde{p}_0 E + c.c.)$ as the buffer-layer wave pressure.

If we define the Fourier transform of w_0 with respect to x_1 to be

$$F(w_0) = \int_{-\infty}^{\infty} w_0(x_1, Y_1, \bar{z}) e^{-i\omega x_1}\, dx_1 \tag{3.38}$$

with real parameter ω, then it follows that

$$F(w_0) = -\frac{F(J_0)}{2(ic_0\rho_0\omega_1/\mu_0)^{1/2}} \exp[-(ic_0\rho_0\omega_1/\mu_0)^{1/2}|Y_1|] \tag{3.39}$$

from (3.35), where $\omega_1 = \omega - i\tau$ and τ is a small real parameter. Inverting (3.39), via the Fourier Convolution Theorem, yields

$$w_0 = -\frac{1}{2\rho_0} \left(\frac{C}{\pi c_0}\right)^{1/2} \int_{-\infty}^{x_1} \frac{J_0(s, \bar{z})}{(x_1 - s)^{1/2}} \exp\left[\frac{-c_0\rho_0^2 Y_1^2}{4C(x_1 - s)}\right] ds \tag{3.40}$$

on use of $\mu_0 = C\rho_0^{-1}$ from the equations of viscosity-temperature and state.

The first non-trivial normal velocity component of the vortex v_2, has the solution

$$v_2 = \left[-d_1 - \frac{\rho_1}{\rho_0}(\bar{a}_2 b_1 - c_2) + \left(\frac{2\rho_1^2}{\rho_0^2} - \frac{\rho_2}{\rho_0}\right)V_0 + \left(\frac{2c_0\rho_1}{\rho_0^2} - \frac{b_1}{\rho_0}\right)\rho_{10} - \frac{c_0\rho_{11}}{\rho_0}\right]\frac{Y_1^2}{2} - \\ -\frac{\partial}{\partial \bar{z}}\int_0^{Y_1} w_0\, \mathrm{d}\hat{Y}_1 - A_4(x_1, \bar{z}) \tag{3.41}$$

upon integrating the appropriate order of the continuity equation. Here $A_4(x_1, \bar{z})$, an arbitrary function, denotes the negative value of v_2 at $Y_1 = 0$ which, owing to the flow structure in the critical layer (Appendix B), is known to be continuous there.

Next, we can determine that

$$\rho_0 c_0 \frac{\partial \bar{u}_4}{\partial x_1} - \mu_0 \frac{\partial^2 \bar{u}_4}{\partial Y_1^2} = \rho_0 b_1 \frac{\partial}{\partial \bar{z}}\int_0^{Y_1} w_0\, \mathrm{d}\hat{Y}_1, \tag{3.42}$$

at the relevant order in the streamwise-momentum equation. Here

$$\bar{u}_4 = u_4 - \frac{b_4 Y_1^4}{24} - (b_3\bar{a}_2 + d_2)\frac{x_1 Y_1^2}{2} - \frac{1}{\rho_0 c_0}\int \bar{A}_4\, \mathrm{d}x_1, \tag{3.43}$$

where $\bar{A}_4$ is a combination of A_4 and linear terms multiplying x_1. The boundary conditions on $\bar{u}_4$ are

$$\bar{u}_4 \to 0 \quad \text{as} \quad x_1 \to -\infty \quad \text{and} \quad \bar{u}_4 \text{ is bounded as } |Y_1| \to \infty. \tag{3.44}$$

Furthermore, we assume that $\bar{u}_4$, $\partial\bar{u}_4/\partial Y_1$ are continuous across $Y_1 = 0$, and this is again based on the critical-layer analysis. The above system for $\bar{u}_4$ together with (3.39) implies

$$F(\bar{u}_4) = \pm\frac{b_1 C}{2c_0^2\rho_0^2\omega_1^2}\frac{\partial}{\partial \bar{z}}[F(J_0)] \times \\ \times\left\{1 - \exp\left(-\left(\frac{ic_0\rho_0^2\omega_1}{C}\right)^{1/2}|Y_1|\right)\left(1 + \frac{1}{2}\left(\frac{ic_0\rho_0^2\omega_1}{C}\right)^{1/2}|Y_1|\right)\right\} \tag{3.45}$$

according as $Y_1 > 0$ or $Y_1 < 0$, in turn. It follows that $\bar{u}_4$, $\partial^2\bar{u}_4/\partial Y_1^2$ are zero at $Y_1 = 0$, while $\partial^2\bar{u}_4/\partial Y_1^2$, $\partial^2\bar{u}_4/\partial Y_1^3$ are continuous and $\partial^4\bar{u}_4/\partial Y_1^4$ is discontinuous across $Y_1 = 0$.

It is worth observing that the wave-induced temperature T_4 satisfies a system similar in type to that of u_4, and in particular

$$\left[\frac{\partial^4 T_4}{\partial Y_1^4}\right]_{Y_1=0^-}^{Y_1=0^+} = \hat{C}\frac{\partial J_0}{\partial \bar{z}}, \tag{3.46}$$

where $\hat{C}$ is constant. Similar jump conditions hold for ρ_4, μ_4, but these functions, along with T_4, bear no significant influence on the interaction and therefore need not be addressed explicitly.

Now we turn our attention to the wave motion. The first few orders yield simple solutions, namely

$$\tilde{p}_0 = p^{(0)}(x_1, \bar{a}_0, \bar{z}), \qquad \tilde{p}^{(1)} = 0, \tag{3.47a,b}$$

where consistent matching with (3.1) has been imposed. Next, we find that

$$\tilde{w}_0 = \frac{i}{\alpha_0 b_1 \rho_0 \gamma M_\infty^2 Y_1} \frac{\partial \tilde{p}_0}{\partial \bar{z}}, \qquad \tilde{v}_0 = \frac{i}{\alpha_0 b_1 \rho_0 \gamma M_\infty^2} \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0 \right), \tag{3.48a,b}$$

$$\tilde{u}_0 = \frac{i}{\alpha_0} \frac{\partial \tilde{w}_0}{\partial \bar{z}}, \qquad \tilde{p}_2 = -\tfrac{1}{2} i \alpha_0 b_1 \rho_0 \gamma M_\infty^2 \tilde{v}_0 Y_1^2 = p^{(1)}(x_1, \bar{a}_0, \bar{z}), \tag{3.48c,d}$$

and

$$\tilde{w}_1 = \frac{i}{\alpha_0 b_1 Y_1} \left(c_0 \frac{\partial \tilde{w}_0}{\partial x_1} - \frac{C}{\rho_0^2} \frac{\partial^2 \tilde{w}_0}{\partial Y_1^2} \right) - \frac{\rho_1}{2\rho_0} Y_1 \tilde{w}_0, \tag{3.49a}$$

$$\tilde{v}_1 = \frac{1}{2} \left(\frac{q_3}{q_2} - \frac{\rho_1}{\rho_0} \right) \tilde{v}_0 Y_1, \qquad \tilde{u}_1 = \frac{i}{\alpha_0} \left(\frac{\partial \tilde{w}_1}{\partial \bar{z}} + \frac{1}{2} \left(\frac{q_3}{q_2} - \frac{\rho_1}{\rho_0} \right) \tilde{v}_0 \right), \tag{3.49b,c}$$

$$\tilde{p}_3 = -i \alpha_0 b_1 \rho_0 \gamma M_\infty^2 q_3 \tilde{v}_0 \frac{Y_1^3}{6 q_2} - \rho_0 c_0 \gamma M_\infty^2 Y_1 \frac{\partial \tilde{v}_0}{\partial x_1} + P_3(x_1, \bar{z}), \tag{3.49d}$$

$$\tilde{\rho}_0 = \frac{i \rho_1}{\alpha_0 b_1 Y_1} \tilde{v}_0, \quad \tilde{T}_0 = -\frac{i \rho_1}{\alpha_0 b_1 \rho_0^2 Y_1} \tilde{v}_0, \quad \tilde{\mu}_0 = C \tilde{T}_0, \tag{3.49e,f,g}$$

where the as yet arbitrary function P_3 is unimportant. It is possible to obtain expressions for $\tilde{w}_2, \tilde{v}_2, \tilde{u}_2, \tilde{p}_4, \tilde{\rho}_1, \tilde{T}_1, \tilde{\mu}_1$, but our main interest is with the equation for $\tilde{p}_5$ which, when solved, will enable us to match with the wave-pressure jump from the core. The equation for $\tilde{p}_5$ is found to be

$$\begin{aligned} \frac{1}{\gamma M_\infty^2} \left(\frac{\partial^2 \tilde{p}_5}{\partial Y_1^2} - \frac{2}{Y_1} \frac{\partial \tilde{p}_5}{\partial Y_1} \right) &= \nu_1 Y_1 \frac{\partial \tilde{v}_0}{\partial x_1} + i \nu_2 Y_1 x_1 \tilde{v}_0 - \\ &\quad -2 i \alpha_0 \rho_0 \left(\tilde{v}_0 \left(\frac{\partial \bar{u}_4}{\partial Y_1} - \frac{1}{Y_1} \bar{u}_4 \right) + \tilde{w}_0 \frac{\partial \bar{u}_4}{\partial \bar{z}} \right) + \tilde{h}(x_1, Y_1, \bar{z}) \end{aligned} \tag{3.50}$$

where

$$\nu_1 = -\frac{\alpha_0 b_1^3 \rho_0^2 M_\infty^2 c_0 b_{-1}}{\gamma_0^2}, \qquad \nu_2 = \frac{\alpha_0 b_1^3 \rho_0^2 M_\infty^2 a_{-1}}{\gamma_0^2}, \tag{3.51a,b}$$

are real and constant-valued and $\tilde{h}$ is a function consisting of a sum of terms of the type $Y_1^n (n \neq 1)$ which has no importance in the ensuing analysis. Integrating (3.50) we obtain

$$\begin{aligned} \frac{1}{Y_1^2} \frac{\partial \tilde{p}_5}{\partial Y_1} - \frac{i \nu_1}{\alpha_0 b_1 \rho_0} \ln|Y_1| \frac{\partial}{\partial x_1} \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0 \right) + \frac{\nu_2}{\alpha_0 b_1 \rho_0} x_1 \ln|Y_1| \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0 \right) \\ = \frac{1}{b_1} \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0 \right) \left[\frac{1}{Y_1^2} \left(\bar{u}_4 - Y_1 \frac{\partial \bar{u}_4}{\partial Y_1} \right) + \int_0^{Y_1} \frac{1}{\hat{Y}_1} \frac{\partial^2 \bar{u}_4}{\partial \hat{Y}_1^2} \, d\hat{Y}_1 \right] - \\ - \frac{1}{b_1} \left(\frac{\partial \tilde{p}_0}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}} \right) \left[\frac{1}{Y_1^2} \left(\bar{u}_4 - Y_1 \frac{\partial \bar{u}_4}{\partial Y_1} \right) - \int_0^{Y_1} \frac{1}{\hat{Y}_1} \frac{\partial^2 \bar{u}_4}{\partial \hat{Y}_1^2} \, d\hat{Y}_1 \right] + \\ + \gamma M_\infty^2 \int_0^{Y_1} \frac{\hat{h}}{\hat{Y}_1^2} \, d\hat{Y}_1 + D^{\pm}(x_1, \bar{z}) \end{aligned} \tag{3.52}$$

according as $Y_1 > 0$ or $Y_1 < 0$, respectively. Here we have used (3.48a,b) to substitute for $\tilde{v}_0, \tilde{w}_0$, whilst $D^{\pm}$ are constants of integration. The term in $\tilde{p}_5$ of order $|Y_1|^3$ is required both as $|Y_1| \to 0$ (to match with the critical layer solution) and as $|Y_1| \to \infty$ (to match with the core solution).

Firstly, as $|Y_1| \to 0$ we use the classical linear critical-layer property

$$\mathrm{FP}\left\{\left[\frac{1}{Y_1^2}\frac{\partial \tilde{p}_5}{\partial Y_1}\right]_{Y_1=0^-}^{Y_1=0^+}\right\} = (i\pi)\frac{i\nu_1}{\alpha_0 b_1 \rho_0}\frac{\partial}{\partial x_1}\left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right)$$
$$-(i\pi)\frac{\nu_2}{\alpha_0 b_1 \rho_0} x_1 \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right), \tag{3.53}$$

i.e. we increase the logarithm in (3.52) by $(i\pi)$ as Y_1 increases from negative to positive [36,37]. It is noted, however, that $b_1 > 0$ must be imposed, for if $b_1 < 0$ then the logarithm would be *decreased* by $(i\pi)$ instead, so that $(i\pi) \to -(i\pi)$ in (3.53). Therefore we see that

$$D^+ - D^- = \frac{-\pi\nu}{\alpha_0 b_1 \rho_0}\frac{\partial}{\partial x_1}\left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right) - \frac{i\pi\nu_2}{\alpha_0 b_1 \rho_0} x_1 \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right). \tag{3.54}$$

Secondly, as $|Y_1| \to \infty$ it follows that

$$\mathrm{FP}\left\{\left(\frac{1}{Y_1^2}\frac{\partial \tilde{p}_5}{\partial Y_1}\right)\Bigg|_{\pm\infty}\right\} = D^{\pm} + \frac{1}{b_1}\left[\left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right) + \frac{\partial \tilde{p}_0}{\partial \bar{z}}\frac{\partial}{\partial \bar{z}}\right]\int_0^{\pm\infty}\frac{1}{\hat{Y}_1}\frac{\partial^2 \bar{u}_4}{\partial \hat{Y}_1^2}\,\mathrm{d}\hat{Y}_1, \tag{3.55}$$

because $\bar{u}_4$ is bounded and $\partial\bar{u}_4/\partial Y_1$ decays exponentially fast, as $|Y_1| \to \infty$. By applying a Fourier transformation to the integral in (3.55), then substituting for $F(\bar{u}_4)$ from (3.45), and finally inverting, we deduce that

$$\int_0^{\pm\infty}\frac{1}{Y_1}\frac{\partial^2 \bar{u}_4}{\partial Y_1^2}\,\mathrm{d}Y_1 = \pm\frac{b_1}{4c_0}\frac{\partial}{\partial \bar{z}}\int_{-\infty}^{x_1} J_0(s, \bar{z})\,\mathrm{d}s. \tag{3.56}$$

The finite-part expression in (3.55) has to equate to

$$\left(\frac{\partial^3 p^{(1)}}{\partial \bar{y}^3}\right)\Bigg|_{\bar{a}-}^{\bar{a}+}$$

through matching with the core solution, where we recall that $p^{(1)}$ as defined in (3.1d) satisfies (3.27) near $\bar{y} = \bar{a}_0$. Hence it follows that

$$\frac{-\pi\nu_1}{\alpha_0 b_1 \rho_0}\frac{\partial}{\partial x_1}\left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right) - \frac{i\pi\nu_2}{\alpha_0 b_1 \rho_0} x_1 \left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right) +$$
$$+\frac{1}{2c_0}\left[\left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right)\frac{\partial}{\partial \bar{z}} + \frac{\partial \tilde{p}_0}{\partial \bar{z}}\frac{\partial^2}{\partial \bar{z}^2}\right]\int_{-\infty}^{x_1} J_0(s, \bar{z})\,\mathrm{d}s$$
$$= M_1^2\left\{x_1 r(x_1)(G_a^+ - G_a^-) + ic_0 r'(x_1)(G_b^+ - G_b^-) - \frac{1}{c_0^2 \rho_w M_\infty^2} Q_w^- r(x_1)\right\}\cos\beta_0 \bar{z}. \tag{3.57}$$

On setting $\tilde{p}_0 = r(x_1)\cos\beta_0\bar{z}$ and further, writing $r(x_1) = \gamma M_\infty^2 \tilde{r}(x_1)$, the coefficients of $\cos\beta_0\bar{z}$ in (3.57) equate to

$$\frac{\pi\nu_1\gamma_0^2}{\alpha_0 b_1\rho_0}\tilde{r}'(x_1) + \frac{i\pi\nu_2\gamma_0^2}{\alpha_0 b_1 \rho_0}x_1\tilde{r}(x_1) + \\ + \frac{\pi(2/3)^{2/3}(-2/3)!}{c_0(\alpha_0 b_1)^{5/3}(C^2/\rho_0)^{2/3}}\beta_0^4(2\beta_0^2-\gamma_0^2)\tilde{r}(x_1)\int_{-\infty}^{x_1}|\tilde{r}^2(s)|\,\mathrm{d}s \\ = M_1^2\left\{x_1\tilde{r}(x_1)(G_a^+ - G_a^-) + ic_0\tilde{r}'(x_1)(G_b^+ - G_b^-) - \frac{1}{c_0^2\rho_w M_\infty^2}Q_w^-\tilde{r}(x_1)\right\}, \tag{3.58}$$

where we have used (3.37) to substitute for J_0. In (3.58) the five terms present correspond respectively to the influences of the critical-layer phase shift, nonlinearity involving the induced vortex feedback, nonparallelism of the basic flow, the core-flow contribution, and the Stokes-layer effect. This equation determines the nonlinear evolution of the wave-amplitude downstream and is examined next.

3.3. Downstream Development of the Interaction

The governing amplitude equation for the wave has the basic form

$$\bar{C}\tilde{r}'(x_1) + \bar{A}\tilde{r}(x_1)\int_{-\infty}^{x_1}|\tilde{r}^2(s)|\,\mathrm{d}s - (\bar{B}x_1 + i\bar{D})\tilde{r}(x_1) = 0, \tag{3.59}$$

where $\bar{A}$ is a real constant, $\bar{B}, \bar{C}$ are complex constants and $\bar{D}$ is a constant which is real provided that a suitable linear transformation in x_1 has been made. (The constant $\bar{D}$ can in fact be taken as zero without loss of generality as in [3]).

This equation is identical in form to that obtained in the work on incompressibility by Smith, Brown and Brown [3], except that here the constant $\bar{B}$ is complex (owing to a_{-1} being non-zero) instead of real as in their paper, and of course we have different values for $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ in general. Consequently, after a few minor modifications, their results regarding the possible downstream developments of the flow interaction apply equally in our case. Here we give a brief description of each possible outcome, further details being given in ref. [3].

The first possibility is that of far-downstream saturation where the wave-pressure amplitude is asymptotic to a finite non-zero constant, i.e.

$$|\tilde{r}(x_1)| \to |\tilde{\pi}_0| \quad \text{as} \quad x_1 \to \infty. \tag{3.60}$$

In this case the interaction matches with the strongly nonlinear interaction for $\bar{x} = O(1)$ described by Hall and Smith [1]. For consistency, the coefficients of (3.59) must be constrained in the manner

$$\bar{A}\tilde{B} > 0 \quad \text{as} \quad \tilde{B}\bar{C}_r < 0, \tag{3.61}$$

where $\tilde{B} = \bar{B}_r + \bar{B}_i\bar{C}_i/\bar{C}_r$ is real and the subscripts r and i denote the real and imaginary parts of a quantity in turn.

Secondly, a singular response may occur at a finite position downstream (say $x_1 = x_1^*$), whereby

$$|\tilde{r}(x_1)| \sim (x_1^* - x_1)^{-1} \quad \text{as} \quad x_1 \to (x_1^*)^-. \tag{3.62}$$

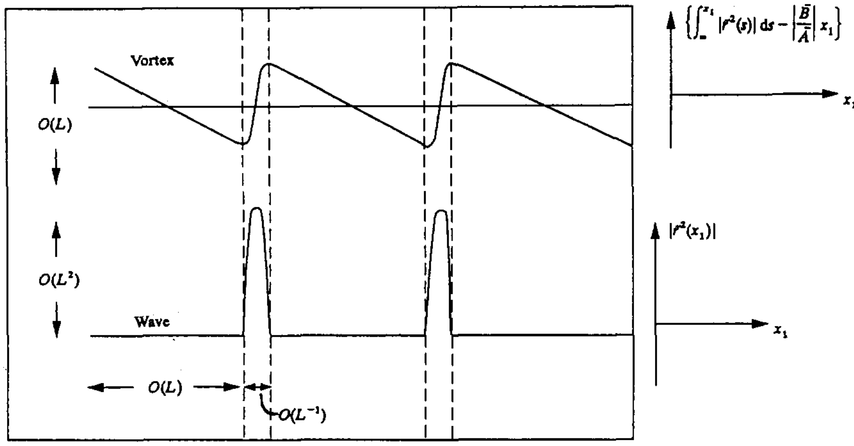

Figure 2. Schematic diagram depicting the behaviour of the vortex and the wave for the large-amplitude periodic solutions outlined in Section 3.2.

At this stage, the buffer layers start to shrink and eventually merge with the critical layer when $x_1^* - x_1 = O(Re^{-1/12})$. The 'new' critical layer is then of a non-equilibrium type wherein streamwise variations counterbalance viscous effects in the governing wave equations. In particular, this alters the calculation of the nonlinear coefficient in the associated wave-amplitude equation. See also, *e.g.*, refs. [29, 35, 38, 39]. Again we require (3.61) to hold in order for this option to be valid.

Thirdly, the waves may decay exponentially fast downstream. The vortex must persist, however, owing to its dependence on the global development of the wave solution overall x_1 and not on its local behaviour, as verified by the integral term J_0. Once more, the associated criteria are given by (3.61).

The fourth and final option is that of periodic or bounded motion, which can be rather more subtle than the others in some circumstances, with the interaction developing a two-phase cycle far downstream. The first phase is similar to the third option, having the waves decaying rapidly to leave pure vortex motion. After a distance of $O(L)$ say (where $L \gg 1$), the brief second phase, of extent $O(L^{-1})$, comes into play. This phase sees the wave-amplitude growing explosively and reactivating full vortex/wave interaction, only to decay again in a similarly fast manner. Figure 2, adapted from [3], describes the behaviour of the vortex and the wave in a little more detail. The conditions to be satisfied by the coefficients here are

$$\bar{A}\tilde{B} > 0 \quad \text{and} \quad \tilde{B}\bar{C}_r > 0. \tag{3.63}$$

The second condition, by the way, corresponds to the nonparallel-flow effects being stabilising for x_1 negative but destabilising for x_1 positive.

We note that other solutions of (3.59) exist but none of these are compatible with the input condition $\tilde{r}(-\infty) = 0$. The above options are discussed further in Section 5.

4. Strongly Compressible Interactions

4.1. General Prandtl number

Here we examine the compressible inflectional-wave/vortex interaction, derived in Section 3 above, in the limit of high Mach number. We initially allow the Prandtl number to be arbitrary but later, for reasons which will become apparent, we apply the restrictions $0 < \sigma < 2$ and $\sigma \neq 0.5$.

The oncoming basic-flow profile is chosen to be the Blasius one, defined by

$$\bar{U}_0 = \frac{\mathrm{d}f}{\mathrm{d}z^*}(z^*), \qquad \bar{p}_0 = 0, \tag{4.1a,b}$$

where f satisfies

$$\frac{\mathrm{d}^3 f}{\mathrm{d}z^{*3}} + \frac{1}{2} f \frac{\mathrm{d}^2 f}{\mathrm{d}z^{*2}} = 0, \tag{4.2}$$

subject to

$$f(0) = \frac{\mathrm{d}f}{\mathrm{d}z^*}(0) = 0, \qquad \frac{\mathrm{d}f}{\mathrm{d}z^*}(\infty) = 1, \tag{4.3a–c}$$

and z^* is the Howarth–Dorodnitsyn variable defined as

$$z^* = \int_0^{\bar{y}} \frac{\mathrm{d}\bar{y}_1}{\bar{T}_0}. \tag{4.4}$$

The basic-flow density is $\bar{\rho}_0 = \bar{T}_0^{-1}$ via the equation of state, while the basic-flow temperature $\bar{T}_0$ satisfies

$$\sigma^{-1} \frac{\mathrm{d}^2 \bar{T}}{\mathrm{d}z^{*2}} + \frac{1}{2} f \frac{\mathrm{d}\bar{T}_0}{\mathrm{d}z^*} + (\gamma - 1) M_\infty^2 \left(\frac{\mathrm{d}^2 f}{\mathrm{d}z^{*2}} \right)^2 = 0, \tag{4.5}$$

subject to

$$\bar{T}_0(0) = T_w, \qquad \bar{T}_0(\infty) = 1 \ \text{(imposed)}, \tag{4.6a,b}$$

from the energy equation.

Although an explicit solution for $\bar{T}_0$ is generally unobtainable (but see 4.2 below), we can determine its asymptotic form, as $z^* \to \infty$, which is important below. In fact two types of asymptote emerge, corresponding to $\sigma < 2$ (when the forcing term in (4.5) is negligible in the far field) and $\sigma \geq 2$ (when the forcing is significant). We focus our attention here on the first case where it is found that

$$\bar{T}_0 \sim 1 + \frac{2kM_\infty^2}{\sigma Z} e^{-\sigma Z^2/4} \tag{4.7}$$

where $Z = z^* - 1.731$ and k is an unknown $O(1)$ constant related to the global solution for $\bar{T}_0$ via the expression

$$k = \lim_{Z \to \infty} \left[\frac{\sigma Z e^{\sigma Z^2/4}}{2M_\infty^2} (\bar{T}_0(Z) - 1) \right]. \tag{4.8}$$

Here the asymptotic expansion

$$f \sim Z + 2\mu_a Z^{-2} e^{-Z^2/4} (1 - 2Z^{-2} + O(Z^{-4})) \tag{4.9}$$

as $Z \to \infty$, where $\mu_a \approx 0.468$ [40], has been used in deducing (4.8).

To establish the critical-layer location (say $Z = Z_s$) we use the inflection-point condition (2.7), which has the equivalent form

$$\bar{T}_0 \frac{\mathrm{d}^2\bar{U}_0}{\mathrm{d}Z^2} = 2\frac{\mathrm{d}\bar{T}_0}{\mathrm{d}Z}\frac{\mathrm{d}\bar{U}_0}{\mathrm{d}Z} \quad \text{at} \quad Z = Z_s. \tag{4.10}$$

This simplifies to

$$\frac{\mathrm{d}\bar{T}_0}{\mathrm{d}Z} + \frac{1}{4}f\bar{T}_0 = 0 \quad \text{at} \quad Z = Z_s, \tag{4.11}$$

using (4.1a) and (4.2). Then, assuming $Z_s \gg 1$, it follows from substitution of (4.7) into (4.11) that

$$\frac{ke^{\sigma Z_s^2/4}}{Z_s} = \frac{\sigma}{2(2\sigma-1)M_\infty^2}, \tag{4.12}$$

which bears the approximate solution

$$Z_s = 2\Gamma/\sigma^{1/2} \tag{4.13}$$

where $\Gamma \approx [\ln M_\infty^2]^{1/2}$. Clearly $\sigma = 1/2$ is not admissible in the light of (4.12) and below, and indeed the restriction $\sigma > 1/2$ is needed to keep Z_s positive. This result also ties in well with Grubin and Trigub's [13,14] analytical work on hypersonic boundary-layer stability, where the same conditions on σ emerge upon seeking the critical-layer location. Notice that the factors $(2\sigma - 1)$ and so on in (4.12) are products of the basic flow features combined with the generalised inflection-point condition.

We now proceed to analyse the flow structure in the vicinity of $Z = Z_s$, where a temperature-adjustment layer of relative thickness $O(\Gamma^{-1})$ is known to exist [9]. We define the local coordinate $\tilde{z}$ by

$$\tilde{z} = \frac{\Gamma}{\sigma^{1/2}}(Z - Z_s) \tag{4.14}$$

and, after a little working, find that

$$\bar{U}_0 \approx 1 + L\Gamma^{(1/\sigma-1)}M_\infty^{-2/\sigma}e^{-\tilde{z}}, \tag{4.15}$$

$$\bar{T}_0 \approx 1 + \frac{1}{(2\sigma-1)}e^{-\sigma\tilde{z}}, \tag{4.16}$$

where

$$L = -\mu_a\left(\frac{\sigma}{2(2\sigma-1)k}\right)^{1/\sigma}\left(\frac{2}{\sigma^{1/2}}\right)^{(1/\sigma-1)} \tag{4.17}$$

is an $O(1)$ constant. Moreover, since $\bar{U}_0|_{Z=Z_s}$ determines the principal wavespeed, we have

$$c_0 \approx 1 + L\Gamma^{(1/\sigma-1)}M_\infty^{-2/\sigma}. \tag{4.18}$$

In passing, it is worth observing that

$$\bar{y} \sim M_\infty^2\bar{J} + Z_s + \bar{Y}/\Gamma \tag{4.19}$$

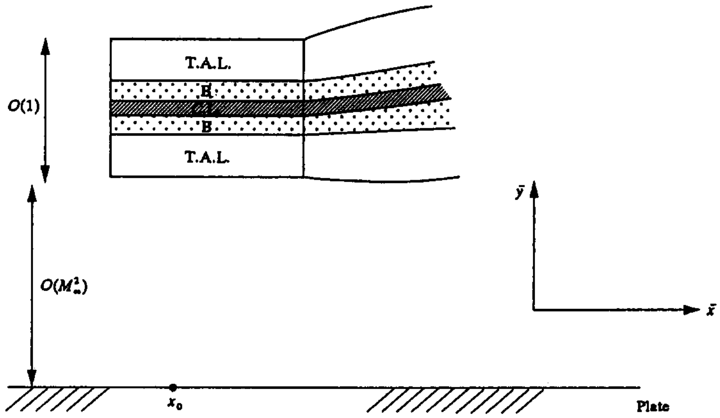

Figure 3. Sketch of the governing flow structure of the wave/vortex interaction for $M_\infty \gg 1$.

in the temperature-adjustment layer (with local coordinate $\bar{Y}$), where

$$\bar{J} = \int_0^\infty (\bar{T}_0/M_\infty^2)\,\mathrm{d}\bar{y} \tag{4.20}$$

is an $O(1)$ constant since $\bar{T}_0 \sim M_\infty^2$ when $\bar{y} = O(1)$. The asymptotic form (4.19) demonstrates that he buffer layers and the critical layer, which are embedded in the relatively thick temperature-adjustment layer, have been shifted by an $O(M_\infty^2)$ amount in $\bar{y}$ from the wall (see Figure 3).

The approximate system for the leading-order wave pressure,

$$\frac{\mathrm{d}^2 P_0}{\mathrm{d}\tilde{z}^2} - \frac{2e^{-\tilde{z}}}{(1-e^{-\tilde{z}})}\frac{\mathrm{d}P_0}{\mathrm{d}\tilde{z}} - \bar{\gamma}_0^2\left(1 + \frac{1}{(2\sigma-1)}e^{-\sigma\tilde{z}}\right)^2 P_0 = 0 \tag{4.21}$$

with

$$P_0(\pm\infty) = 0, \tag{4.22a,b}$$

arises from (3.11), (3.12a,b) after the transformation to $\tilde{z}$-space, with

$$(\alpha_0, \beta_0, \gamma_0) = \frac{\Gamma}{\sigma^{1/2}}(\bar{\alpha}_0, \bar{\beta}_0, \bar{\gamma}_0) + O(\Gamma^{-1}) \tag{4.23a–c}$$

here characterising the wavenumber expansions associated with the vorticity mode [9]. An investigation of this system by asymptotic analysis reveals that the wave pressure decays exponentially fast at the outer edge of the temperature-adjustment layer (as $\tilde{z} \to \infty$) and doubly exponentially fast at the lower edge (as $\tilde{z} \to -\infty$); this agrees with the findings of Smith and Brown [9] for unit Prandtl number where, moreover, the authors obtained an explicit solution for P_0 (see also 4.2 below).

The higher-order mean-flow and density terms $\bar{U}_1$ and $\bar{\rho}_1$ follow quite simply through examination of the global structure of the Blasius flow. We write

$$\bar{U} = \bar{U}(\eta), \qquad \bar{\rho} = \bar{\rho}(\eta) \tag{4.24a,b}$$

where

$$\eta = \frac{z_1^*(\bar{x}, \bar{y})}{(2C\bar{x})^{1/2}} \tag{4.25}$$

is the global similarity variable and

$$z_1^*(\bar{x},\bar{y}) = \int_0^{\bar{y}} \frac{1}{\bar{T}(\eta)}\,\mathrm{d}\bar{y}_1 \qquad (= z^*(\bar{y}) + O(\bar{x}-\bar{x}_0), \quad \text{as} \quad \bar{x} \to \bar{x}_0). \tag{4.26}$$

Moreover, we impose $(C\bar{x}_0)^{1/2} = 1$ so that $\eta \to z^*/2$ as $\bar{x} \to \bar{x}_0$. Thus, because

$$\bar{\rho}_1 = \lim_{\bar{x}\to\bar{x}_0} \left(\frac{\partial}{\partial \bar{x}}\,\bar{\rho}(\eta)\right) \tag{4.27}$$

and

$$\frac{\partial}{\partial \bar{x}} \to \frac{\partial}{\partial \bar{x}} + \left[\frac{\partial z_1^*/\partial \bar{x}}{(2C\bar{x})^{1/2}} - \frac{\eta}{2\bar{x}}\right]\frac{\partial}{\partial \eta}, \tag{4.28}$$

it follows that

$$\bar{\rho}_1 = -\tfrac{1}{2}C\bar{y}\bar{\rho}_{0\bar{y}}. \tag{4.29}$$

By a similar argument we may show that

$$\bar{U}_1 = -\tfrac{1}{2}C\bar{y}\bar{U}_{0\bar{y}}. \tag{4.30}$$

An important aim now is to determine the form for the wave-amplitude equation as $M_\infty \to \infty$. There are at least two such forms, as discussed successively below. We examine each contribution in the equation starting with the external non-parallel term. Specifically we are interested in the quantity $(G_a^+ - G_a^-)$, as defined in (3.31a), and a little work shows that

$$\begin{aligned} G_a^+ - G_a^- \approx\; & \tfrac{1}{2}C\,\mathrm{FP}\int_0^\infty \left(\frac{P_{0\bar{y}}^2}{\bar{M}^2} - \left(\frac{\gamma_0^2}{\bar{M}^2} - \alpha_0^2\right)P_0^2\right)\mathrm{d}\bar{y} \\ & -2c_2\,\mathrm{FP}\int_0^\infty \frac{\bar{U}_{o\bar{y}}}{(\bar{U}_0 - c_0)^2\bar{M}^2}\,P_0P_{0\bar{y}}\,\mathrm{d}\bar{y}, \end{aligned} \tag{4.31}$$

where the properties (4.29), (4.30) have been used in the derivation. The first integral in (4.31) is negligible provided we have $|c_2| \gg \Gamma^{(1/\sigma-1)}M_\infty^{-2/\sigma}$. Therefore, with the dominant part of the second integral lying in the range $(\bar{a}_0^-, \bar{a}_0^+)$ (owing to the relative smallness of the wave pressure outside the temperature-adjustment layer), (4.31) simplifies to

$$G_a^+ - G_a^- \approx 2c_2\int_{\bar{a}_0^-}^{\bar{a}_0^+} \frac{\bar{U}_{0\bar{y}}}{(\bar{U}_0 - c_0)^2\bar{M}^2}\,P_0P_{0\bar{y}}\,\mathrm{d}\bar{y}. \tag{4.32}$$

It is convenient here to introduce the transformations $s = e^{-\bar{z}}$, where we find that

$$G_a^+ - G_a^- \approx \frac{c_2(2\sigma-1)^3}{4\sigma^5 b_1^3}\,\Gamma^4 M_\infty^{-2}\int_0^\infty \frac{s}{(1-s)^4}\,P_0P_{0s}\,\mathrm{d}s. \tag{4.33}$$

The properties

$$\bar{U}_0 - c_0 = \frac{b_1\sigma^{1/2}}{\Gamma\rho_0}(1-s), \qquad \bar{T}_0 = 1 + \frac{1}{(2\sigma-1)}\,s^\sigma \tag{4.34a,b}$$

have been used in the above derivation, where

$$b_1 = -\frac{L(2\sigma - 1)}{2\sigma^{3/2}} \Gamma^{1/\sigma} M_\infty^{-2/\sigma} \tag{4.35a}$$

and

$$\rho_0 = \frac{(2\sigma - 1)}{2\sigma} \tag{4.35b}$$

Under the new transformation, the wave pressure P_0 satisfies

$$s^2 P_{0ss} + \frac{s(1+s)}{(1-s)} P_{0s} - \bar{\gamma}_0^2 \left(1 + \frac{s^\sigma}{(2\sigma - 1)}\right)^2 P_0 = 0, \tag{4.36}$$

subject to

$$P_0(0) = P_0(\infty) = 0. \tag{4.37a,b}$$

Unfortunately this system does not appear to yield an explicit solution for P_0 for general values of σ, unlike the model-fluid case later.

Next, the quantity involving $(G_b^+ - G_b^-)$ in (3.58) is found to reduce to

$$G_b^+ - G_b^- \approx \frac{(2\sigma - 1)^3}{4\sigma^{9/2}\bar{\alpha}_0 b_1^3} \Gamma^3 M_\infty^{-2} \int_0^\infty \frac{s}{(1-s)^4} P_0 P_{0s} \, \mathrm{d}s. \tag{4.38}$$

We now consider the linear critical-layer growth effects from (3.58) and in particular, the coefficients ν_1, ν_2 defined in (3.51a,b) above, where dominant balances imply that

$$\nu_1 \approx -\frac{(2\sigma - 1)^4}{16\sigma^4} \Gamma^2, \qquad \nu_2 \approx \frac{(2\sigma - 1)^4}{16\sigma^{9/2}} \alpha_0 c_2 \Gamma^2. \tag{4.39a,b}$$

The Stokes layer effect (proportional to Q_w^-) can be seen to be negligible, as a result of the relatively tiny wave pressure outside the temperature-adjustment layer.

The balancing of the linear and non-parallel contributions forces the interaction coordinate x_1 to grow with Mach number in the fashion

$$x_1 \sim \Gamma^{-1/2} |c_2|^{-1/2}. \tag{4.40}$$

Finally to preserve the nonlinear interaction it necessarily follows that

$$|\tilde{r}| \sim \Gamma^{(1/3\sigma - 1/6)} M_\infty^{-2/3\sigma} |c_2|^{1/2}, \tag{4.41}$$

by balancing the nonlinear contribution with, for example, the linear critical-layer term.

Writing $c_2 = \delta \tilde{c}_2$, where $\tilde{c}_2$ is $O(1)$ and $\delta \gg \Gamma^{(1/\sigma - 1)} M_\infty^{-2/\sigma}$, we rescale $\tilde{r}, x_1$ in the manner

$$\tilde{r} = \Gamma^{(1/3\sigma - 1/6)} M_\infty^{-2/3\sigma} \delta^{1/2} \bar{r}(\bar{x}_1) + \cdots, \tag{4.42}$$

$$x_1 \equiv \Gamma^{-1/2} \delta^{-1/2} \bar{x}_1, \tag{4.43}$$

and then the leading-order balances in (3.58) equate to

$$\left(1+\frac{i(2\sigma-1)H}{\pi\sigma^2\bar{\gamma}_0^2}\right)\left[\frac{\mathrm{d}\bar{r}}{\mathrm{d}\bar{x}_1}-\frac{i\bar{\alpha}_0\bar{c}_2}{\sigma^{1/2}}\,\bar{x}_1\bar{r}(\bar{x}_1)\right]$$
$$=\left\{\frac{8\sigma^{5/3}(2/3)^{2/3}(-2/3)!\bar{\gamma}_0^4}{(\bar{\alpha}_0L)^{2/3}(2\sigma-1)^3C^{4/3}}\left(1-\frac{\bar{\alpha}_0^2}{\bar{\gamma}_0^2}\right)^2\left(1-\frac{2\bar{\alpha}_0^2}{\bar{\gamma}_0^2}\right)\right\}\bar{r}(\bar{x}_1)\int_{-\infty}^{\bar{x}_1}|\bar{r}^2(u)|\,\mathrm{d}u. \tag{4.44}$$

Here

$$H=\int_0^\infty \frac{s}{(1-s)^4}\,P_0P_{0s}\,\mathrm{d}s, \tag{4.45}$$

is a real integral which depends exclusively on σ. To analyse the associated flow behaviour downstream we write

$$\bar{C}\left[\frac{\mathrm{d}\bar{r}}{\mathrm{d}\bar{x}_1}-\frac{i\bar{\alpha}_0c_2}{\sigma^{1/2}}\,\bar{x}_1\bar{r}\right]+\bar{A}\bar{r}\int_{-\infty}^{\bar{x}_1}|\bar{r}^2(u)|\,\mathrm{d}u=0, \tag{4.46}$$

where

$$\bar{C}=-1-\frac{i(2\sigma-1)H}{\pi\sigma^2\bar{\gamma}_0^2}, \tag{4.47}$$

$$\bar{A}=\frac{8\sigma^{5/3}(2/3)^{(2/3)}(-2/3)!\bar{\gamma}_0^4}{(\bar{\alpha}_0L)^{2/3}(2\sigma-1)^3C^{4/3}}\left(1-\frac{\bar{\alpha}_0^2}{\bar{\gamma}_0^2}\right)^2\left(1-\frac{2\bar{\alpha}_0^2}{\bar{\gamma}_0^2}\right). \tag{4.48}$$

Setting $\bar{r}=\bar{R}(\bar{x}_1)\exp[i\bar{\theta}(x_1)]$, with $\bar{R},\bar{\theta}$ real functions, leads to a phase equation for $\bar{\theta}$ and the following modulus equation,

$$\frac{\mathrm{d}\bar{R}}{\mathrm{d}\bar{x}_1}+\left(\frac{\bar{A}}{\bar{K}}\right)\bar{R}\int_{-\infty}^{\bar{x}_1}\bar{R}^2\,\mathrm{d}u=0, \tag{4.49}$$

where $\bar{K}=|\bar{C}|^2/\bar{C}_r$ is real. This yields the simple result

$$\bar{R}=\left(-\frac{\bar{K}}{\bar{A}}\right)^{1/2}(\bar{x}_s-\bar{x}_1)^{-1}, \tag{4.50}$$

suggesting that the flow solution becomes singular as $\bar{x}_1\to\bar{x}_s^-$ downstream. Furthermore, since $\bar{R}$ is real, we require $(\bar{K}/\bar{A})<0$ as a necessary constraint which, given the values of $\bar{A}$ and $\bar{C}$, would force the wave angle to lie in the interval $(\pi/4,\pi/2)$. This result would appear to be quite restrictive.

The above is one form for the interaction at large M_∞. A second form follows from further scrutiny, indicating that the flow interaction is governed by a more generalised solution upstream, and that (4.44) merely represents one of several possible transition paths. The corresponding scalings are found to be

$$x_1=\Gamma^{-1/2\sigma}M_\infty^{1/\sigma}\hat{x}_1, \tag{4.51}$$

$$\tilde{r}=\Gamma^{(5/6\sigma-2/3)}M_\infty^{-5/3\sigma}\hat{r}(\hat{x}_1)+\cdots, \tag{4.52}$$

and, upon setting $\hat{r} = \hat{R}(\hat{x}_1)\exp[i\hat{\theta}(\hat{x}_1)]$, the modulus equation is

$$\hat{C}\frac{\mathrm{d}\hat{R}}{\mathrm{d}\hat{x}_1} + \bar{A}\hat{R}\int_{-\infty}^{\hat{x}_1}\hat{R}^2(s)\,\mathrm{d}s - \tilde{B}\hat{x}_1\hat{R} = 0. \tag{4.53}$$

There is in fact little simplification so far from the original equation. Also here

$$\tilde{B} = -\frac{C\bar{\alpha}_0 L(2\sigma-1)^2}{4\pi\sigma^{5/2}}\left(\frac{H_1}{\bar{\gamma}_0^2} + \frac{6\sigma^2\bar{q}_3}{(2\sigma-1)^2}\right) \tag{4.54}$$

denotes the real quantity defined in Section 3.3 above, while

$$H_1 = \int_0^\infty \frac{1}{s(1-s)^2}\left[s^2 P_{0s}^2 - \bar{\gamma}_0^2\left(1 + \frac{s^\sigma}{(2\sigma-1)}\right)^2 P_0^2\right]\mathrm{d}s, \tag{4.55}$$

and

$$\bar{q}_3 = 1 - \frac{(2\sigma-1)^2}{6\sigma^2\bar{\gamma}^2}\left[\left(\frac{\mathrm{d}^3 P_0}{\mathrm{d}s^3}\right)_{s=1}\right]. \tag{4.56}$$

Because of the complicated coefficients still involved it is a considerable task to obtain numerical solutions describing the complete wave/vortex interaction under given experimental conditions. Thus, even before addressing (4.53) above, we must first solve the system for the wave pressure in the temperature-adjustment layer to determine H, H_1 and $\bar{q}_3$ as well as obtain the eigenrelation that determines $\bar{\gamma}_0$, and then solve the basic-flow temperature system in the entire boundary layer to determine L. However, the special case of a model fluid, i.e. a Chapman fluid with unit Prandtl number, addressed next, is found to yield a considerably simplified analysis, with both $P_0, \bar{T}_0$ shown to have explicit solutions.

4.2. The Model Fluid

Here we re-consider the analysis derived in 4.1 above, in the special case of unit Prandtl number. A summary of the important results is now given.

Firstly, we note that the oncoming Blasius flow as defined in (4.1)–(4.4) is independent of the Prandtl number, in contrast with the basic-flow temperature system (4.5), (4.6a,b) which, for $\sigma = 1$, bears the solution

$$\bar{T}_0 = 1 + \frac{(\gamma-1)}{2}M_\infty^2 U_0(1-U_0) + (T_w - 1)(1-U_0), \tag{4.57}$$

a result sometimes referred to as Crocco's Law. Given (4.57) we are able to evaluate the constants k, L defined in (4.8), (4.17) respectively. It is found that

$$k = \frac{1}{2}\phi\mu_a(\gamma-1) \quad \text{and} \quad L = -\frac{1}{\phi(\gamma-1)} \tag{4.58a,b}$$

where

$$\phi \equiv \frac{1}{2} + \frac{(T_w-1)}{(\gamma-1)M_\infty^2} \tag{4.59}$$

is a linear transformation of the wall temperature.

Next, we address the wave-pressure system in the temperature-adjustment layer, given by (4.21), (4.22a,b). For unit Prandtl number this system is known to yield the neutral solution

$$P_0 = s^{1/4}e^{(1-s)/4} \tag{4.60}$$

with the corresponding dispersion relation [9].

$$\bar{\gamma}_0 = \frac{1}{4} \quad \text{or} \quad \bar{\alpha}_0^2 + \bar{\beta}_0^2 = \frac{1}{16} \tag{4.61}$$

Subsequently, with P_0 known explicitly, the terms $H, H_1, \bar{q}_3$ defined in (4.45), (4.55), (4.56) follow as

$$H = -\frac{1}{16}I, \quad H_1 = \frac{1}{8}(2e\pi)^{1/2}, \quad \bar{q}_3 = -\frac{1}{3}, \tag{4.62a–c}$$

with

$$I \equiv \int_0^\infty \frac{s^{-1/2}}{(s-1)}\, e^{(1-s)/2}\, \mathrm{d}s. \tag{4.63}$$

Upon substituting the above values for $L_1, H, H_1, \bar{q}_3, \bar{\gamma}_0$, and otherwise setting $\sigma = 1$ in (4.53), we may deduce that the governing wave-modulus equation is

$$(1 + I^2/\pi^2)\,\frac{\mathrm{d}\hat{R}}{\mathrm{d}\bar{x}_1} + \left(\frac{C\bar{\alpha}_0}{2\pi(\gamma-1)\phi}\,[(2e\pi)^{1/2} - 1]\right)\bar{x}_1\hat{R}(\bar{x}_1)$$
$$= \left\{\frac{(2/3)^{2/3}(-2/3)!\phi^{2/3}(\gamma-1)^{2/3}(1-16\bar{\alpha}_0^2)^2(1-32\bar{\alpha}_0^2)}{32\bar{\alpha}_0^{2/3}C^{4/3}}\right\}\hat{R}(\bar{x}_1)\int_{-\infty}^{\bar{x}_1} |\hat{R}^2(u)|\,\mathrm{d}u \tag{4.64}$$

in this model fluid case.

Expressed in terms of (3.59) above, we have the coefficients being

$$\bar{C} = Ii/\pi - 1, \qquad \tilde{B} = \frac{C\bar{\alpha}_0}{2\pi(\gamma-1)\phi}\,[(2e\pi)^{1/2} - 1], \tag{4.65a,b}$$

$$\bar{A} = \frac{(2/3)^{2/3}(-2/3)!\phi^{2/3}(\gamma-1)^{2/3}(1-16\bar{\alpha}_0^2)^2(1-32\bar{\alpha}_0^2)}{32\bar{\alpha}_0^{2/3}C^{4/3}}. \tag{4.65c}$$

It was mentioned in Section 3.3 above that the condition $\bar{A}\tilde{B} > 0$ has to be satisfied for each of the four possible types of downstream behaviour. With $\bar{\alpha}_0$ assumed to be positive, it follows that

$$(1 - 32\bar{\alpha}_0^2)\phi^{-1/3} > 0 \tag{4.66}$$

must hold. Next, with $\bar{C}_r = -1$ from (4.65a), we observe that

$$(\tilde{B}\bar{C}_r)\phi < 0. \tag{4.67}$$

Consequently, two distinct cases emerge corresponding to the sign of ϕ. Firstly, for $\phi > 0$ (or $T_w > 1 - (\gamma-1)M_\infty^2/2$ from (4.59)), we have $\tilde{B}\bar{C}_r < 0$ so that any one of far-downstream saturation, finite-distance singularity or exponentially decaying waves may occur and furthermore, from (4.61), (4.66), the wave angle arctan $(\bar{\beta}_0/\bar{\alpha}_0)$ is confined to the interval $(\pi/4, \pi/2)$. Secondly, for $\phi < 0$ (or $T_w < 1-(\gamma-1)M_\infty^2/2$), we have $\tilde{B}\bar{C}_r > 0$ so that only the option of periodicity may occur with the wave angle now lying in the interval $(0, \pi/4)$.

5. Final Comments

In the present study on inflectional-wave/vortex interaction in boundary layers we have considered two distinct regimes of compressibility, namely when $M_\infty = O(1)$ (covering the subsonic and supersonic ranges) and when $M_\infty \gg 1$ (covering the hypersonic range). In each case, the interaction is found to be governed effectively by a nonlinear integro-differential equation for the wave pressure, and this has been shown to yield four kinds of downstream behaviour which we address once more here. The first option has the flow solution saturating far downstream, with the wave-pressure amplitude asymptoting to a constant non-zero value. Moreover, in the limit $x_1 \to O(Re^{1/4})$, the interaction develops into the strongly nonlinear type described by Hall and Smith [1]; see also Smith, Brown and Brown [3]. Secondly, an algebraic singularity may occur at a finite position downstream. At the approach to this position, the buffer layers begin to shrink until finally, at a distance $O(Re^{-1/12})$ away, they merge with the critical layer. When this happens, at larger amplitudes, the critical layer becomes of a non-equilibrium type and, as shown by Wu, Lee and Cowley [38] in shear layers for example, the nonlinear mechanics is significantly changed. As regards a next stage, we might expect a continuation into an algebraic singularity possibly leading to stronger Euler-type interactions [19] or, instead, the wave-pressure amplitude may be damped to a zero or non-zero finite value. A full examination of the wave/vortex compressible interaction in this regime would appear worthwhile and efforts may be directed towards this in the future. Thirdly, the wave pressure may die out in an exponentially fast manner to leave, in effect, pure vortex motion downstream. This feature, which has been found to occur in a number of previous studies on non-linear wave/vortex interaction [28,41], demonstrates that the interaction can serve to alter the mean flow to a stable one comprising longitudinal vortices. The fourth and final option seems at first sight a rather curious one. It involves a periodic response which may consist of long quiescent regions of predominantly vortex flow, interspersed with brief eruptions of wave/vortex interaction. It is interesting that the criteria associated with this option (see Section 3.3) are different from the first three, making it possible to predict its occurrence through calculation of the interaction coefficients alone. This would be particularly useful when seeking computational results, since the third and fourth options may be virtually indistinguishable until the first 'burst' appears some way downstream in the latter option. The above options are identical in nature to those found in the related incompressible work of Smith, Brown and Brown [3], indicating that the basic form of the wave/vortex interaction remains intact at non-large Mach numbers.

Concerning the hypersonic theory of Section 4, one of our current goals is to identify the Mach-number regime for which this theory first breaks down as M_∞ continues to increase. Several possibilities present themselves. It was thought first that the relevant regime is $M_\infty = O[Re^{\sigma/[4(3\sigma+1)]}]$, corresponding to the streamwise variations of the critical layer altering the flow structure within. Further scrutiny however suggested that these effects would not significantly alter the flow interaction. Next it was thought that the basic flow should become fully non-parallel in the temperature-adjustment layer when $M_\infty \sim Re^{\sigma/[4(2\sigma+1)]}$, but that turned out not to be so and this option also was dismissed. Our current belief is that there are two possible outcomes depending exclusively on the size of the Prandtl number, specifically whether σ is greater than or less than 2/3. If $\sigma > 2/3$, it is believed that the hypersonic-interaction regime involving external shock-layer influence for which $M_\infty = O(Re^{1/6})$ enters [34,40,42]. On the other hand if $\sigma < 2/3$, it is believed that the new regime enters when $M_\infty \sim Re^{\sigma/4}$ (neglecting powers of ln Re) and this corresponds to the buffer layers merging with the critical layer and/or the temperature-adjustment layer. However, this latter outcome would preclude dealing with air flow, for which the Prandtl number is approximately 0.72, and so perhaps the former case is the more relevant one. We should reiterate here that our work in the hypersonic range has featured a pair of vorticity waves. However, alternative problems involving a pair of acoustic waves, or even having one acoustic wave and one vorticity wave, could produce interesting results and hopefully will be addressed in future research.

Further computational work on the hypersonic interaction theory of Section 4 could be helpful, with a view to evaluating the interaction coefficients for various values of wave angle, Prandtl number and wall temperature. The comparatively simple form and subsequent evaluation of the interaction coefficients in the model-fluid case of Section 4.2 should provide a useful check on the accuracy of any general numerical scheme used.

Direct practical questions resulting from this work concern three issues: first, whether the wall conditions can be altered to favourably affect hypersonic transition [(3.58) tends to suggest surface cooling could be beneficial]; second, whether the theory applies also for axisymmetric base flows (the short scales suggest it does); and third, where precisely does the hypersonic transition originating at the edge of the boundary layer eventually influence the wall properties to a substantial extent? These require further theoretical research. Other possible extensions to the present work would include analysing the hypersonic problem for Prandtl numbers σ near 1/2 (where the current flow solution becomes singular, heralding a change in the decay behaviour near the free stream) and $\sigma \geq 2$, adapting the current theories to the general case of Sutherland's viscosity-temperature law, adding a small amount of cross-flow, or having a non-uniform pressure gradient, caused by an uneven surface for example.

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Appendix A. The Stokes Layer

This layer, of relative thickness $O(\epsilon^3)$, is located immediately adjacent to the wall. It bears some influence on the wave dynamics in the core through the term Q_w^- of (3.26), which we now formally derive.

Writing $\bar{y} = \epsilon^3\hat{y}$, the flow variables expand as

$$\bar{u} = \epsilon^3\lambda_w\hat{y} + \cdots + \epsilon^7\hat{u}E + \cdots, \tag{A1a}$$

$$\bar{v} = \cdots + \epsilon^4\hat{v}E + \cdots, \tag{A1b}$$

$$\bar{w} = \epsilon\hat{w}E + \cdots, \tag{A1c}$$

$$\bar{p} = 1 + \cdots + (\epsilon^7\hat{p}^{(0)} + \epsilon^{10}\hat{p}^{(1)} + \epsilon^{13}\hat{p})E + \cdots, \tag{A1d}$$

$$\bar{\rho} = \rho_w + \cdots + \epsilon^7\hat{\rho}E + \cdots, \tag{A1e}$$

$$\bar{\mu} = \mu_w + \cdots + \epsilon^7 \hat{\mu} E + \cdots, \tag{A1f}$$

$$\bar{T} = T_w + \cdots + \epsilon^7 \hat{T} E + \cdots, \tag{A1g}$$

where the terms multiplying E are the wave contributions and the remainder denote mean-flow and small vortex quantities.

The equations governing the wave motion are found to be

$$-i\alpha_0 c_0 \hat{\rho} + i\alpha_0 \rho_w \hat{u} + \rho_w \hat{v}_{\hat{y}} + \rho_w \hat{w}_{\bar{z}} = 0, \tag{A2a}$$

$$-i\alpha_0 c_0 \rho_w [\hat{u}, \hat{w}] = -\frac{1}{\gamma M_\infty^2} \left[i\alpha_0, \frac{\partial}{\partial \bar{z}} \right] \hat{p}^{(0)} + \mu_w \frac{\partial^2}{\partial \hat{y}^2} [\hat{u}, \hat{w}], \tag{A2b,c}$$

$$\hat{p}_{\hat{y}}^{(0)} = 0, \quad \hat{p}_{\hat{y}}^{(1)} = 0, \quad -i\alpha_0 c_0 \rho_w \hat{v} = -\frac{1}{\gamma M_\infty^2} \hat{p}_{\hat{y}} + \mu_w \hat{v}_{\hat{y}\hat{y}}, \tag{A2d–f}$$

$$\hat{\rho} T_w + \rho_w \hat{T} = \hat{p}^{(0)}, \tag{A2g}$$

$$-i\alpha_0 c_0 \rho_w \hat{T} = \sigma^{-1} \frac{\partial}{\partial \hat{y}} \left(\mu_w \frac{\partial \hat{T}}{\partial \hat{y}} \right) - i\alpha_0 c_0 \frac{(\gamma - 1)}{\gamma} \hat{p}^{(0)}. \tag{A2h}$$

These equations, in conjunction with the boundary conditions $\hat{u} = \hat{v} = \hat{w} = 0, \hat{T} = \hat{T}_0$ (say) at $\hat{y} = 0$ and matching with the core-flow solution as $\hat{y} \to \infty$, yield

$$[\hat{u}, \hat{w}] = \frac{1}{\alpha_0 c_0 \rho_w \gamma M_\infty^2} (1 - e^{-m\hat{y}}) \left[\alpha_0, -i \frac{\partial}{\partial \bar{z}} \right] \hat{p}^{(0)} \tag{A3a,b}$$

$$\hat{p}^{(0)} = p^{(0)}(x_1, 0, \bar{z}), \qquad \hat{p}^{(1)} = p^{(1)}(x_1, 0, \bar{z}) \tag{A3c,d}$$

$$\hat{T} = \hat{T}_0 e^{-m_1 \hat{y}} + \frac{(\gamma - 1)}{\rho_w \gamma} \hat{p}^{(0)} (1 - e^{-m_1 \hat{y}}), \tag{A3e}$$

$$\hat{\rho} = (\hat{p}^{(0)} - \rho_w \hat{T}) \rho_w, \tag{A3f}$$

and

$$\hat{v} = \frac{i}{\alpha_0 c_0 \rho_w \gamma M_\infty^2} \left(\frac{\partial^2 \hat{p}^{(0)}}{\partial \bar{z}^2} - \alpha_0^2 \hat{p}^{(0)} \right) \left[\hat{y} + \frac{1}{m} (e^{-m\hat{y}} - 1) \right] + $$
$$+ i\alpha_0 c_0 \left(\frac{1}{m_1} (\rho_w \hat{T}_0 - \hat{p}^{(0)}) (e^{-m_1 \hat{y}} - 1) + \frac{\hat{p}^{(0)}}{\gamma} \left[\hat{y} + \frac{1}{m_1} (e^{-m_1 \hat{y}} - 1) \right] \right), \tag{A3g}$$

where the exponential factors

$$m \equiv (-i\alpha_0 c_0 / C)^{1/2} \rho_w, \qquad m_1 \equiv \sigma^{1/2} m \tag{A4a,b}$$

are both complex constants. Here, in addition, the mean-flow properties $\mu_w = CT_w, T_w = \rho_w^{-1}$, from the viscosity-temperature and state equations have been applied. It now follows from (A2f) and (A3g) that

$$\hat{p}_{\hat{y}} \sim \left\{ \gamma_0^2 \left(\hat{y} - \frac{1}{m} \right) - \gamma M_\infty^2 \alpha_0^2 c_0^2 \rho_w \left(\frac{\hat{y}}{\gamma} - \frac{1}{m_1} \left[\left(\frac{1}{\gamma} - 1 \right) + \rho_w \hat{T}_w \right] \right) \right\} \hat{P}_w r(x_1) \cos \beta_0 \bar{z} \tag{A5}$$

as $\hat{y} \to \infty$, where we have written

$$\hat{p}^{(0)} = \hat{P}_w r(x_1)\cos\beta_0\bar{z}, \qquad \hat{T}_0 = \hat{P}_w\hat{T}_w r(x_1)\cos\beta_0\bar{z} \tag{A6a,b}$$

and $\hat{P}_w(\equiv P_0(0))$ and $\hat{T}_w$ are constants. Hence $p^{(1)}$, as defined in (3.1d) above, must satisfy the inner boundary condition

$$\left(\frac{\partial p^{(1)}}{\partial\bar{y}}\right)_{\bar{y}=0} = \left[-\gamma_0^2 + \frac{\alpha_0^2 c_0^2 \rho_w^2 \gamma M_\infty^2}{\sigma^{1/2}}\left(\hat{T}_w - \frac{1}{\rho_w}\frac{(\gamma-1)}{\gamma}\right)\right] m^{-1}\hat{P}_w r(x_1)\cos\beta_0\bar{z}, \tag{A7}$$

so that

$$Q_w^- = \left[-\frac{\gamma_0^2}{\rho_w} + \frac{\alpha_0^2 c_0^2 \rho_w \gamma M_\infty^2}{\sigma^{1/2}}\left(\hat{T}_w - \frac{1}{\rho_w}\frac{(\gamma-1)}{\gamma}\right)\right]\hat{P}_w\left(\frac{-i\alpha_0 c_0}{C}\right)^{-1/2} \tag{A8}$$

immediately follows.

6. Appendix B. The Critical Layer

Two main results are sought through analysing the flow structure in the critical layer. Firstly we require the phase difference $D^+(x_1\bar{z}) - D^-(x_1,\bar{z})$ in (3.52). The work needed to evaluate this quantity from first principles is considerable; however, a much easier method exists, based on the classical argument that $(D^+ - D^-)$ equates to $(\pm i\pi)$ multiplied by the coefficient of $\ln|Y_1|$ in the asymptotic expansion of $Y_1^{-2}\tilde{p}_{5Y_1}$ as $|Y_1| \to 0$, where $\tilde{p}_5$ is defined in (3.33d) (see e.g. Lin [36]). The choice of sign depends on which logarithmic branch is taken as Y_1 increases from negative to positive via the complex plane; classical theory has proved that the positive sign is always taken when the basic-flow critical-layer shear, b_1 is positive, while the negative sign is taken if b_1 is negative. Hence, If we assume that $b_1 > 0$ then (3.54) follows. The second result obtained here is the evaluation of $J_0(x_1,\bar{z})$, the wave-amplitude-squared forcing of the vortex. Details of this are now given.

The buffer-layer analysis of 3.2 infers the following expansions as $Y_1 \to \epsilon^{1/2}Y$, where Y is the $O(1)$ critical-layer coordinate:

$$\begin{aligned}\bar{u} = {} & c_0 + \epsilon^2 b_1 Y + \epsilon^3 c_2 x_1 + \epsilon^4 b_2 Y^2/2 + \\ & +\epsilon^5\tilde{U}_0 + \epsilon^6\tilde{U}_1 + \cdots + \epsilon^7\tilde{U}_2 + \cdots + \epsilon^8\tilde{U}_3 + \cdots,\end{aligned} \tag{B1a}$$

$$\bar{v} = V_0 + \epsilon\tilde{V}_2 + \epsilon^2\tilde{V}_3 + \cdots + \epsilon^3\tilde{V}_4 + \cdots + \epsilon^4\tilde{V}_5 + \cdots, \tag{B1b}$$

$$\bar{w} = \epsilon^{-1}\tilde{W}_0 + \tilde{W}_1 + \cdots + \epsilon\tilde{W}_2 + \cdots + \epsilon^2\tilde{W}_3 + \cdots, \tag{B1c}$$

$$\bar{p} = 1 + \epsilon^3 p_0 x_1 + \epsilon^6 p_1 x_1^1/2 + \epsilon^7\tilde{P}_2 + \epsilon^8\tilde{P}_3 = \cdots, \tag{B1d}$$

$$\bar{\rho} = \rho_0 + \epsilon^2\rho_1 Y + \epsilon^3(\rho_1\bar{a}_2 + \rho_{10})x_1 + \epsilon^4\rho_2 Y^2/2 + \cdots, \tag{B1e}$$

$$\bar{\mu} = \mu_0 + \epsilon^2\mu_1 Y + \epsilon^3(\mu_1\bar{a}_2 + \mu_{10})x_1 + \epsilon^4\mu_2 Y^2/2 + \cdots, \tag{B1f}$$

$$\bar{T} = T_0 + \epsilon^2 T_1 Y + \epsilon^3(T_1\bar{a}_2 + T_{10})x_1 + \epsilon^4 T_2 Y^2/2 + \cdots. \tag{B1g}$$

Here it is understood that the terms capped by a tilde () comprise both wave and vortex/mean-flow contributions, and hence may be written as $\tilde{U}_n = u_{nE}E + u_{nE}^*E^{-1} + u_{nN}$ for $n \geq o$, and similarly for $\tilde{V}_n, \tilde{W}_n$, etc. Both the wave component multiplying E and the mean component are independent of X and $\bar{t}_1$ until $n \geq 3$, when the latter has to include contributions proportional to E^2 and E^{-2}. The expansions (B1a–g) also contain odd powers of $\epsilon^{1/2}$ (not shown) which are forced by the buffer layer. These terms do not affect the value of J_0 however, and are therefore not addressed here.

The important leading-order wave solutions are

$$w_{0E} = -\frac{1}{\mu_0 \gamma M_\infty^2}\left(\frac{\alpha_0 b_1 \rho_0}{\mu_0}\right)^{-2/3} \frac{\partial \tilde{p}_0}{\partial \bar{z}} \int_0^\infty \exp\left[-i\left(\frac{\alpha_0 b_1 \rho_0}{\mu_0}\right)^{1/3} Yt - \frac{t^3}{3}\right] dt,$$

$$v_{2E} = \frac{i}{\alpha_0 b_1 \rho_0 \gamma M_\infty^2}\left(\frac{\partial^2 \tilde{p}_0}{\partial \bar{z}^2} - \alpha_0^2 \tilde{p}_0\right),$$

$$u_{0E} = \frac{i}{\alpha_0}\frac{\partial w_{0E}}{\partial \bar{z}}, \qquad p_{2E} = \tilde{p}_0(x_1, \bar{z}), \tag{B2a–d}$$

where consistent buffer-layer matching (as $|Y| \to \infty$) has been applied.

For the vortex/mean-flow, we find that simple solutions hod at the first three orders. These are

$$p_{2N} = v_{2N} = w_{0N} = 0, \qquad u_{0N} + \left(d_1 - \frac{b_1 \rho_1 \bar{a}_2}{\rho_0}\right) x_1 Y; \tag{B3a–d}$$

$$p_{3N} = w_{1N} = 0, \qquad u_{1N} = b_1 Y^3/6 + u_4(x_1, 0, \bar{z}),$$

$$v_{3N} = \left(\bar{a}_2 b_1 - c_2 - \frac{\rho_1 V_0{}'}{\rho_0} - \frac{\rho_{10} c_0}{\rho_0}\right) Y; \tag{B4a–d}$$

and

$$p_{4N} = x_1^3 p_2/6, \qquad w_{2N} = 0,$$

$$u_{2N} = (b_3 \bar{a}_2 + d_2) x_1 Y^2/2 + \hat{K}(x_1, \bar{z}), \qquad v_{4N} = -A_4(x_1, \bar{z}), \tag{B5a–d}$$

successively, where $\hat{K}$ is an unknown constant, p_2 is the local cubic derivative of the basic-flow pressure and matching with the buffer layers holds as $|Y| \to \infty$. Additionally we note the solutions

$$\mu_0 = CT_0, \qquad T_0 = \rho_0^{-1}, \tag{B6a,b}$$

linking ρ_0, μ_0 and T_0. At the next order, we need only concern ourselves with the spanwise-momentum equation for w_{3N} which, when integrated between $(-\infty, \infty)$, determines J_0. It is

$$\rho_0\left[i\alpha_0(u_{0E}^* w_{0E} - u_{0E} w_{0E}^*) + v_{2E}\frac{\partial w_{0E}^*}{\partial Y} + v_{2E}^*\frac{\partial w_{0E}}{\partial Y} + w_{0E}\frac{\partial w_{0E}^*}{\partial \bar{z}} + w_{0E}^*\frac{\partial w_{0E}}{\partial \bar{z}}\right]$$
$$= \mu_0 \frac{\partial^2 w_{3N}}{\partial Y^2}, \tag{B7}$$

which leads to

$$J_0 \equiv \left[\frac{\partial w_{3N}}{\partial Y}\right]_{Y\to-\infty}^{Y\to\infty} = \frac{\rho_0}{\mu_0}\int_{-\infty}^{\infty}\left(2\frac{\partial}{\partial \bar{z}}|w_{0E}|^2 + v_{2E}\frac{\partial w_{0E}^*}{\partial Y} + v_{2E}^*\frac{\partial w_{0E}}{\partial Y}\right)\mathrm{d}Y, \tag{B8}$$

using (B2c) to substitute for u_{0E}. Further substitution of (B2a,b) for v_{2E}, w^{2E} respectively, and subsequent manipulation of the integral yields

$$J_0 = \frac{2\pi(2/3)^{2/3}(-2/3)!}{(\gamma M_\infty^2)^2(\alpha_0 b_1)^{(5/3)}(C^2/\rho_0)^{2/3}}\frac{\partial}{\partial \bar{z}}\left(\left|\frac{\partial \tilde{p}_0}{\partial \bar{z}}\right|^2\right). \tag{B9}$$

Here $(-2/3)!$ denotes the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \tag{B10}$$

evaluated at $x = 1/3$.

It is worthwhile mentioning here that in the Hall–Smith theory (where $\bar{x} = O(1)$), a jump term equivalent to (B9) exists. This was calculated in the incompressible regime by Hall and Smith, but later J.W. Elliott (personal communication) determined the compressible value to be

$$J = \frac{\psi}{(\gamma M_\infty^2)^2}\left\{\frac{\partial}{\partial \bar{z}}\left(\left|\frac{\partial \tilde{p}}{\partial \bar{z}}\right|^2\right) - \frac{1}{3}\left[\frac{5b_{\bar{z}}}{b} + \left(2 - \frac{T(\partial\mu/\partial T)}{\mu}\right)\frac{\rho_{\bar{z}}}{\rho} + \frac{11 f_{\bar{z}} f_{\bar{z}\bar{z}}}{(1+f_{\bar{z}}^2)}\right]\left|\frac{\partial \tilde{p}}{\partial \bar{z}}\right|^2\right\}, \tag{B11}$$

where b, ρ, μ, T denote the values of the global basic-flow quantities $\bar{U}_{\bar{y}}, \bar{\rho}, \bar{\mu}, \bar{T}$ at $\bar{y} = f(\bar{x}, \bar{z})$, and $\tilde{p}(\bar{x}, \bar{z})$ denotes the global buffer-layer wave pressure. Also

$$\psi(\bar{x}, \bar{z}) = \frac{2\pi(2/3)^{2/3}(-2/3)!}{[(\alpha b)^5\rho^2\mu^4(1+f_{\bar{z}}^2)^{10}]^{5/3}} \tag{B12}$$

and viscosity is a general function of temperature. Encouragingly, the result matches to (B9) as $\bar{x} \to 0$ in the special case where Chapman's viscosity-temperature law holds. This, in turn, shows that (B9) retains its fundamental form from $x_1 = O(1)$ through to $x_1 = O(\epsilon^{-3})$, and hence the wave-wave forcing is essentially invariant in nature within this range.

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